Abstract

This paper is concerned with the time integration of the equations of motion describing the non-linear dynamic response of beams. Desirable characteristics of integration schemes for structural dynamics problems include unconditional stability, accuracy, and high frequency numerical dissipation. Several schemes exist that present all these features when applied to linear problems. Though the application of those schemes to non-linear problems is often successful, proofs of unconditional stability are rarely derived. A finite difference integration scheme is derived in this paper for the non-linear dynamic response of beams. Though of a finite difference nature, the proposed scheme mimics the integration scheme obtained by applying the time discontinuous Galerkin methodology to a single degree of freedom linear oscillator. A formal proof of unconditional stability for the non-linear problem is derived based on an energy decay argument. Numerical examples using the proposed scheme are given, and the results are compared with the predictions of other available schemes. The accuracy of the proposed scheme and its high frequency numerical dissipation characteristics are demonstrated in these examples.

1. Introduction

This paper is concerned with the time integration of the equations of motion describing the non-linear dynamic response of beams. The main focus is on the derivation of an algorithm presenting high frequency numerical dissipation, and for which unconditional stability can be proven in the non-linear case. An energy decay argument will be used to establish stability [1].

The Newmark algorithm [2] is widely used in structural dynamics. In particular, the average acceleration method, also known as the trapezoidal rule, is an unconditionally stable, second-order accurate scheme when applied to linear problems. The classical stability analysis of this scheme can be readily found in text books [3] and shows that the spectral radius remains exactly equal to unity at all frequencies. An alternate way of proving stability is based on an energy argument. Indeed, it is easily shown that the average acceleration scheme exactly preserves the total energy of the system [1].

For large finite element discretizations, very high frequencies are present in the model and high frequency numerical dissipation is desirable, if not indispensible. Numerical dissipation cannot be introduced in the Newmark method without degrading the accuracy. Hilber et al. [4] introduced the \( \alpha \)-method to remedy this situation. More recently, the generalized-\( \alpha \) method [5] was introduced which achieves high frequency dissipation while minimizing unwanted low frequency dissipation. Both methods have been successfully used for both linear and non-linear problems, though unconditional stability can only be proved for linear systems.
Simo and his co-workers have introduced energy preserving algorithms for non-linear dynamics including rigid body dynamics [6], and elastodynamics [7]. The unconditional stability of these schemes stems from a proof of preservation of the total energy of the system. An energy-preserving scheme for non-linear elastic multi-body systems was proposed by Bauchau [8]. In this scheme, the equations of motion are discretized so that they imply conservation of the total energy for the elastic components of the system, whereas the forces of constraint associated with the kinematic constraints are discretized so that the work they perform vanishes exactly. The combination of these two features of the discretization guarantees the stability of the numerical integration process for non-linear elastic multi-body systems.

Though energy-preserving schemes perform well, their lack of high frequency numerical dissipation can be a problem [8]. First, the time histories of internal forces and velocities can present a very significant high frequency content. Second, it seems that the presence of high frequency oscillations can hinder the convergence process for the solution of the non-linear equations of motion. This was observed in several examples where the dynamic response of the system does involve significant high frequency content. The selection of a smaller time step does not necessarily help this convergence process, as a smaller time step allows even higher frequency oscillations to be present in the response. Finally, it seems that the presence of high frequency oscillations also renders strict energy preservation difficult to obtain. This could prove to be a real limitation of energy-preserving schemes when applied to more and more complex models. For such models, the use of integration schemes presenting high frequency numerical dissipation become increasingly desirable.

It appears that the development of 'energy decaying' schemes, i.e. schemes eliminating the energy associated with vibratory motions at high frequency, is desirable. This is particularly important when dealing with large scale problems, or with problems presenting a complex dynamic response such as constrained multi-body problems.

The key to the development of an energy-decaying scheme is the derivation of an energy decay inequality [1] rather than the discrete energy-conservation law which is central to energy-preserving schemes. A methodology that can systematically lead to an energy-decay inequality is the time discontinuous Galerkin method [9] which was initially developed for hyperbolic equations. Hughes and Hulbert [10, 11] have investigated the use of the time discontinuous Galerkin methodology for elastodynamics. They point out that: 'classical elastodynamics can be converted to first-order symmetric hyperbolic form, which has proved useful in theoretical studies. Finite element methods for first-order symmetric hyperbolic system are thus immediately applicable. However, there seems to be several disadvantages: in symmetric hyperbolic form the state vector consists of displacements, velocities, and stresses which is computationally uneconomical; and the generalization to non-linear elastodynamics seems possible only in special circumstances.' Indeed, writing the non-linear equations of motion of beams in this symmetric hyperbolic form does not appear to be possible.

In this paper an alternate, somewhat hybrid, route is taken. In a first phase, a standard time discontinuous Galerkin method is used to model a single degree of freedom linear oscillator. The equations of motion of this simple system can be readily cast into the symmetric hyperbolic form, and the energy decay inequality follows from the theory of the time discontinuous Galerkin method applied to hyperbolic conservation laws [12, 13]. However, the resulting discretized equations of motion can also be viewed as a finite difference scheme, and moreover, the energy decay inequality can be alternatively obtained by a direct computation of the work done by the discretized inertial and elastic forces over a time step. In other words, the time discontinuous Galerkin formulation is used to obtain a finite difference scheme which can then be proven to imply an energy-decaying inequality without resorting to the formalism of the time discontinuous Galerkin method.

In a second phase, the non-linear equations of motion of beams are discretized in a manner that mimics the finite difference scheme obtained for the linear oscillator. An energy-decay inequality is then derived by a direct computation of the work done by the discretized inertial and elastic forces over a time step, mimicking once more the process used for linear oscillator.

The advantages of proceeding in this fashion are numerous. First, the non-linear equations of motion need not be recast in the symmetric hyperbolic form, which does not seem possible in this case. Second, the state vector only consists of displacements and velocities. Furthermore, the velocities can be readily eliminated resulting in a set of equations solely involving displacements. Third, there is no need to
numerically perform the time integrals that appear in the time discontinuous Galerkin formulation. Finally, a rigorous proof of unconditional stability for the non-linear system is implied by the energy-decay inequality. Numerical experimentation indicates that the salient features of the time discontinuous Galerkin method are inherited by the proposed scheme, in particular, unconditional stability for the non-linear system, and high frequency numerical dissipation with asymptotic annihilation.

The paper is organized in the following manner. The basic time discontinuous Galerkin approximation for a single degree of freedom linear oscillator is presented in Section 2.1, together with an alternate proof of stability based on a direct computation of the work performed by the inertial and elastic forces (Section 2.2). The non-linear equations of motion of beams are presented in Section 3.1, followed by the description of the proposed scheme and a derivation of the energy decay inequality (Section 3.2). Two numerical examples are presented in Section 4; conclusions are presented in the last section.

2. Time discontinuous Galerkin approximation for a single degree of freedom system

2.1. Time discontinuous Galerkin scheme

Consider a linear, single degree of freedom (dof) spring mass system, of which the equation of motion is given as

\[ m\ddot{u}(t) + ku(t) = F(t), \]

where \( u(t) \) is the displacement of the mass \( m \), \( k \) the spring stiffness, \( F(t) \) the excitation force, and \( (\cdot)' \) denotes a time derivative. The time discontinuous Galerkin scheme applied to this system is discussed in [14]. Introducing the momentum \( p = m\dot{u} \) leads to the following two equations which are in the symmetric hyperbolic form:

\[ \dot{p} + ku = F; \quad \dot{u} - \frac{p}{m} = 0. \]

A time discontinuous Galerkin approximation of these equations between the initial and final times \( t_n \) and \( t_{n+1} \) (see Fig. 1), respectively, writes

\[ \int_{t_n}^{t_{n+1}} \left\{ w_1 \left( \dot{u} - \frac{p}{m} \right) + w_2 (\dot{p} + ku - F) \right\} \, dt + w_{1n}^+ (u_n^+ - u_n^-) + w_{2n}^+ (p_n^+ - p_n^-) = 0 \]

(2)

where \( w_1 \) and \( w_2 \) are test functions, and the notations \( (\cdot)^- \), \( (\cdot)^+ \) and \( (\cdot)^{n+1} \) are used to indicate the corresponding quantities at \( t_n \), \( t_n^+ \) and \( t_{n+1}^+ \), respectively.

Integration Eq. (2) by parts yields:

\[ \int_{t_n}^{t_{n+1}} \left( -\dot{w}_1 u - w_1 \frac{p}{m} - w_2 p + w_2 ku - w_2 F \right) \, dt + w_{1n+1}^+ u_{n+1}^- + w_{2n+1}^+ p_{n+1}^- - w_{1n}^- u_n^- - w_{2n}^- p_n^- = 0. \]

(3)

Fig. 1. Typical time step with function discontinuities at boundaries.
A linear in time approximation over the time step is used to discrete the unknowns \( u \) and \( p \), the test functions \( w_1 \) and \( w_2 \), and the excitation force \( F \). The resulting discretized equations of motion are readily found by integrating (3) to find

\[
\frac{m \ddot{u}_{n+1}^+ + \ddot{u}_{n}^-}{2} - \frac{m \ddot{u}_{n}^-}{2} + \frac{2u_{n+1}^+ + u_{n}^+}{6} = \frac{F_{n}^+ + F_{n+1}^-}{6};
\]

\[
\frac{m \ddot{u}_{n+1}^- + \ddot{u}_{n}^+}{2} - \frac{m \ddot{u}_{n}^+}{2} + \frac{2u_{n+1}^- + u_{n}^-}{6} = \frac{F_{n}^- + F_{n+1}^+}{6};
\]

\[
\frac{u_{n+1}^+ + u_{n}^+}{2} - \frac{u_{n}^-}{2} = \frac{u_{n+1}^- + 2u_{n}^-}{6};
\]

\[
\frac{u_{n+1}^- - u_{n}^-}{2} = \frac{2u_{n+1}^- + \dot{u}_{n}^-}{6}.
\]

These four equations may be combined in a linear fashion to yield

\[
\frac{m \dot{u}_{n+1}^+ - m \dot{u}_{n}^-}{\Delta t} + k \frac{u_{n+1}^+ - 2u_{n}^+}{3} = \frac{F_{n}^+ + F_{n+1}^-}{6}; \tag{4}
\]

\[
\frac{m \dot{u}_{n}^+ - m \dot{u}_{n}^-}{\Delta t} + k \frac{u_{n}^-}{6} = \frac{F_{n}^- + F_{n+1}^+}{6}; \tag{5}
\]

\[
\frac{u_{n+1}^+ - u_{n}^+}{2} = \frac{u_{n+1}^- + \dot{u}_{n}^-}{2}; \tag{6}
\]

\[
\frac{u_{n}^-}{2} = \frac{u_{n+1}^- - \dot{u}_{n}^-}{6}. \tag{7}
\]

Eqs. (4)-(7) define a time discontinuous approximation of the equation of motion of the system, Eq. (1). The unconditional stability of the scheme can be proved based on the energy decay inequality that follows from the theory of the time discontinuous Galerkin method applied to hyperbolic conservation laws [12, 13].

This can be confirmed by a conventional analysis of the scheme based on the characteristics of the amplification matrix. The period elongation is \( \Delta T / T = (\omega^2 \Delta t^2) / 270 + O(\omega^6 \Delta t^6) \), while the algorithmic damping is \( \zeta = (\omega^3 \Delta t^3) / 72 + O(\omega^4 \Delta t^4) \), where \( \omega^2 = k / m \). Hence, the scheme is third-order accurate. The spectral radius, period elongation, and algorithmic damping are shown in Figs. 2–4 as functions of \( \Delta t / T = \omega \Delta t / (2\pi) \). The results are compared with generalized-\( \alpha \) method [5] with three different values of spectral radius at infinity, \( \rho_s = 0.9, 0.5 \) and 0.0. Fig. 3 indicates that the time discontinuous Galerkin scheme has better period elongation characteristics than generalized-\( \alpha \) method, while Fig. 4 shows that its low frequency numerical dissipation characteristics are similar to those of the generalized-\( \alpha \) method with \( \rho_s = 0.5 \). Asymptotic annihilation is obtained with the time discontinuous Galerkin scheme. The scheme is unconditionally stable since the spectral radius is always smaller than unity.

2.2. Stability proof based on an energy argument

An alternate way of proving the unconditional stability is based on a direct computation of the work done by the inertial and elastic forces which will be shown to imply an energy decay inequality. The total energy of the system is \( E(u) = K(u) + V(u) \), where the kinetic energy is \( K(u) = \frac{1}{2} \dot{u}^2 \) and the potential energy \( V(u) = \frac{1}{2} ku^2 \). The change in total energy over a time step can be evaluated by computing the work done by the inertial and elastic forces. The discretized equation of motion (4) is multiplied by a displacement increment to yield
where $\Delta W_m$ is the work done by the applied loads. With the help of Eq. (6), this can be rewritten as

$$\left( \frac{u_{n+1}^- - u_n^-}{\Delta t} \right) \frac{m u_{n+1}^- - m u_n^-}{\Delta t} + \left( \frac{u_{n+1}^- - u_n^-}{\Delta t} \right) k \frac{u_{n+1}^- + u_n^-}{2} = \Delta W_m \left( \frac{u_n^+ - u_n^-}{\Delta t} \right)$$

and simplifies to

$$E(u_{n+1}^-) - E(u_n^-) + E(u_n^+ - u_n^-) + \frac{1}{2} (\dot{u}_n^+ - \dot{u}_n^-) m (\ddot{u}_{n+1}^- - \ddot{u}_n^-) + \frac{1}{2} (u_{n+1}^- - u_n^-) k (u_{n+1}^- - u_n^-) = \Delta W_m.$$

Next, the discretized equation of motion (5) is multiplied a displacement increment across the jump to find

$$\left( \frac{u_{n+1}^+ - u_{n+1}^-}{\Delta t} \right) \frac{m u_{n+1}^- - m u_n^-}{\Delta t} + \left( \frac{u_{n+1}^+ - u_{n+1}^-}{\Delta t} \right) k \frac{u_{n+1}^+ - u_{n+1}^-}{6} = \Delta W_{ij} \left( \frac{u_n^+ - u_n^-}{\Delta t} \right),$$

where $\Delta W_{ij}$ is the work done by the applied loads. Using Eq. (7), the above equation rewrites as
and simplifies to
\[-\frac{1}{6}(\dot{u}_n^+ - \dot{u}_n^-)m(\dot{u}_{n+1}^+ - \dot{u}_{n+1}^-) - \frac{1}{6}(u_n^+ - u_n^-)k(u_{n+1}^+ - u_{n+1}^-) = \Delta W_j.\] (9)

Finally, a linear combination of Eqs. (8) and (9) yields the following energy statement:
\[E(u_{n+1}^+) - E(u_{n-1}^-) + E(u_{n+1}^- - u_n^-) = \Delta W_m + 3\Delta W_j.\] (10)

In the absence of externally applied loads, the following energy decay characteristic of the scheme is established:
\[E(u_{n+1}^+) = E(u_{n-1}^-) - E(u_{n+1}^- - u_n^-), \quad \Rightarrow \quad E(u_{n+1}^+) \leq E(u_n^-).\] (11)

This energy decay inequality provides an alternate proof of the unconditional stability of the scheme. Eq. (11) implies exact preservation of energy when the energy associated with the jump \((E(u_{n-1}^- - u_n^-))\) is zero. This can be achieved by enforcing continuity of displacement and momentum at the inter element boundary. In that case the (unconditionally stable) average acceleration scheme is recovered.

3. An energy decaying scheme for beams

3.1. Equations of motion

Let \(\tilde{t}_1, \tilde{t}_2,\) and \(\tilde{t}_3\) form an inertial triad denoted \(\mathcal{I}; \tilde{e}_{01}, \tilde{e}_{02},\) and \(\tilde{e}_{03}\) be a triad denoted \(\mathcal{I}_0\) attached to the reference line of the beam before deformation; and \(\tilde{e}_1, \tilde{e}_2,\) and \(\tilde{e}_3\) a triad denoted \(\mathcal{I}_*\) attached to the reference line of the beam after deformation. The planes formed by \(\tilde{e}_{02}, \tilde{e}_{03},\) and \(\tilde{e}_2, \tilde{e}_3\) define the plane of the cross-section of the beam before, and after deformation, respectively (see Fig. 5).

The strain energy of the beam is written as
\[V = \frac{1}{2} \int_0^L C^* e^* \, dx_1,\] (12)

where \(L\) is the length of the beam and \(x_1\) the curvilinear coordinate along the reference line. \(C^*\) are the components of the sectional stiffness tensor and \(e^*\) the components of the sectional strain vector, both measured in \(\mathcal{I}_*.\) These sectional strains are related to the sectional displacements as
\[e^* = \begin{bmatrix} e_x^* \\
2 \end{bmatrix} = \mathcal{R}^T e = \mathcal{R}^T \left| \begin{array}{c} \varepsilon \\
k \end{array} \right| = \mathcal{R}^T \left| \begin{array}{c} (\alpha' + \frac{u'}{k}) \\
k \end{array} \right| \mathcal{R}_{03}^T \mathbf{1};\] (13)

Fig. 5. Beam reference line in undeformed and deformed position.
where
\[
\mathcal{R} = \begin{bmatrix} R \mathbf{R}_0 & 0 \\ 0 & R \mathbf{R}_0 \end{bmatrix},
\]
and where \( \mathbf{1}^T = [1 \ 0 \ 0] \). The following quantities are all measured in \( \mathcal{F} \): \( R \) defines the components of the rotation tensor from \( \mathcal{F} \) to \( \mathcal{F}_0 \); \( \mathbf{R} \) the components of the rotation tensor from \( \mathcal{F}_0 \) to \( \mathcal{F}^* \); \( \mathbb{u} \) the components of the displacement vector of the deformed reference line with respect to the undeformed configuration; and \( \mathbf{k} \) the components of the sectional elastic curvature vector, with \( \mathbf{k} = \mathbf{R}' \mathbf{R}^T \). The notation \((\cdot)'\) is used to denote a derivative with respect to \( x_1 \), and \((\cdot)\) denotes a skew-symmetric matrix.

The notation \((\cdot)'\) is used to denote a derivative with respect to \( x_1 \), and \((\cdot)\) denotes a skew-symmetric matrix.

The kinetic energy of the beam is expressed as
\[
K = \frac{1}{2} \int_0^L \mathbf{v}^*^T \mathbf{M}^* \mathbf{v}^* \, dx_1,
\]

where \( \mathbf{M}^* \) are the components of the sectional inertia tensor, and \( \mathbf{v}^* \) the components of the sectional velocity vector, both measured in \( \mathcal{F}^* \). These sectional velocities are related to the sectional displacements as
\[
\mathbf{v}^* = \mathcal{R}^T \mathbf{v} = \mathcal{R}^T \begin{bmatrix} \mathbf{u} \\ \mathbf{\omega} \end{bmatrix},
\]

where \( \mathbf{\omega} \) are the components of the sectional angular velocity, with \( \mathbf{\omega} = \mathbf{R} \mathbf{R}^T \); and a superposed dot denotes derivative with respect to time.

Virtual variations in sectional strains and velocities result in:
\[
\delta \mathbf{e}^* = \delta \mathbf{d}^T \mathcal{R} + \delta \mathbf{d}^T \mathbf{U} \left[ (\mathbf{\bar{u}}_0 + \mathbf{\bar{u}})' \right] \mathcal{R},
\]

and
\[
\delta \mathbf{v}^* = \delta \mathbf{d}^T \mathcal{R} + \delta \mathbf{d}^T \mathbf{U} \left[ \mathbf{\bar{u}}^T \right] \mathcal{R},
\]

where \( \mathbf{\delta d}^T = (\delta \mathbf{u}^T, \delta \mathbf{\psi}^T) \) are the virtual displacements and rotations measured in \( \mathcal{F} \). The virtual rotation is defined as \( \delta \mathbf{\psi} = \delta \mathbf{R} \mathbf{R}^T \), and
\[
\mathbf{U} [:] = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}.
\]

The equations of motion of the beam can be obtained from Hamilton's Principle that states:
\[
\int_{t_i}^{t_f} \left( \delta K - \delta V + \delta W \right) \, dt = 0,
\]

where \( \delta W \) is the virtual work done by the externally applied forces. Introducing Eqs. (12) and (15) into Eq. (20) yields:
\[
\int_{t_i}^{t_f} \left( \delta \mathbf{v}^*^T \mathbf{M}^* \mathbf{v}^* - \delta \mathbf{e}^*^T \mathbf{C}^* \mathbf{e}^*^T + \delta W \right) \, dt = 0.
\]

Finally, the equations of motion are found by introducing Eqs. (17) and (18) into Eq. (21) to find:
\[
(\mathcal{R} \mathbf{p}^*) + \mathbf{U} \left[ \mathbf{\bar{u}} \right] \mathcal{R} \mathbf{p}^* - (\mathbf{\dot{f}}^* \left. \right|) - \mathbf{U} \left[ \mathbf{\bar{u}}^\prime \right] \mathbf{\dot{f}}^* = \mathbf{q},
\]

where the sectional momenta \( \mathbf{p}^* = \mathbf{M}^* \mathbf{v}^* \) and forces \( \mathbf{f}^* = \mathbf{C}^* \mathbf{e}^* \) measured in \( \mathcal{F}^* \) were defined. Though these equations of motions can be written in a conservation form, it does not seem possible to recast them in the symmetric hyperbolic form.
3.2. Energy-decaying scheme

An energy-decaying scheme for beams is obtained by extending the single degree of freedom time discontinuous Galerkin method described in Section 2.1 to the equations of motion of a beam, Eq. (22). Discretization (C1) is applied first over the time step, i.e. from \( t_n \) to \( t_{n+1} \):

\[
\frac{\mathbf{p}_{n+1}^{*,*} - \mathbf{p}_n^{*,*}}{\Delta t} + \mathbf{u}_m \left[ \frac{u_{n+1}^{*} - u_n^{*}}{\Delta t} + \frac{2R_mR_0}{4 - a_0} \right] \mathbf{p}_n^{*,*} + \frac{\mathbf{p}_n^{*,*}}{2} - (\mathcal{M} f^*_m)' - \mathbf{u}_m \left[ \frac{\mathbf{u}_0^{*,*} + \mathbf{u}_m^{*,*}}{2} \right] \mathcal{M} f^*_m = q_m,
\]

where \( a \) are the components of the conformal rotation vector of the rotation from \( \mathcal{S}_n \) to \( \mathcal{S}_{n+1} \), measured in \( \mathcal{S} \) (see Appendix A), the subscript \( m \) refers to the time step mid-point, and \( f_m \) refers to elastic forces at this mid-point. Discretization (C1) is then applied across the jump, i.e. from \( t_n \) to \( t_j \):

\[
\frac{\mathbf{R}^*_m \mathbf{p}_n^{*,*} - \mathbf{R}^*_m \mathbf{p}_n^{*,*}}{\Delta t} + \mathbf{u}_m \left[ \frac{u_{n+1}^{*} - u_n^{*}}{\Delta t} - \frac{2R_mR_0}{4 - b_0} \right] \mathbf{p}_n^{*,*} - \frac{\mathbf{p}_n^{*,*}}{2} - (\mathcal{M} f_j)' - \mathbf{u}_m \left[ \frac{\mathbf{u}_0^{*,*} + \mathbf{u}_m^{*,*}}{2} \right] \mathcal{M} f_j = q_j,
\]

where \( b \) are the components of the conformal rotation vector of the rotation from \( \mathcal{S}_n \) to \( \mathcal{S}_j \), measured in \( \mathcal{S} \), the subscript \( j \) refers to the jump 'mid-point', and \( f_j \) refers to elastic forces at this mid-point.

The unconditional stability of the scheme defined by Eqs. (23) and (24) will be established based on an energy argument. The proof follows the steps described in Section 2.2 for the single degree of freedom system. This work done by the inertial and elastic forces in Eq. (23) was computed in Appendix C as

\[
\int_0^L v_m^{*T} M^*(v_m^{*+} - v_m^{*-T}) + e_m^{*T} C^*(e_m^{*+} - e_m^{*-T}) \, dx = \Delta W_m.
\]

By analogy to Eq. (6), the midpoint sectional velocities and strains are selected as:

\[
v_m^* = \frac{v_m^{*+} + v_m^{*-}}{2}, \quad e_m^* = \frac{e_m^{*+} + e_m^{*-}}{2}.
\]

With this choice, Eq. (25) now becomes

\[
E(u_n^{*+}) - E(u_n^{*-}) + \frac{1}{2} \int_0^L (n_m^{*+T} - n_m^{*-T}) M^*(n_m^{*-T} - n_m^{*+T}) + (e_m^{*+T} - e_m^{*-T}) C^*(e_m^{*+T} - e_m^{*-T}) \, dx = \Delta W_m.
\]

This equation is equivalent to (8).

This work done by the inertial and elastic forces in Eq. (24) was computed in Appendix C as

\[
\int_0^L v_j^{*T} M^*(v_j^{*-} - v_j^{*+T}) + e_j^{*T} C^*(e_j^{*-} + e_j^{*+}) \, dx = \Delta W_j.
\]

By analogy to Eq. (7), the midpoint sectional velocities and strains are now selected as

\[
v_j^* = -\frac{v_j^{*-} + v_j^{*+}}{6}, \quad e_j^* = -\frac{e_j^{*-} + e_j^{*+}}{6}.
\]

With this choice, Eq. (28) now becomes

\[
-\frac{1}{6} \int_0^L (v_j^{*+T} - v_j^{*-T}) M^*(v_j^{*-} - v_j^{*+T}) + (e_j^{*+T} - e_j^{*-T}) C^*(e_j^{*-} + e_j^{*+T}) \, dx = \Delta W_j.
\]

This equation is equivalent to (9). Finally, a linear combination of Eqs. (27) and (30) yields the following energy statement:
\[ E(u_{n+1}^-) - E(u_n^+) + E(u_n^+ - u_n^-) = \Delta W + 3 \Delta W_i , \]

which is identical to (10). In the absence of externally applied loads the following energy-decay characteristic of the scheme is established:
\[ E(u_{n+1}^-) = E(u_n^+) - E(u_n^+ - u_n^-) \Rightarrow E(u_{n+1}^-) \leq E(u_n^-) . \]  
This proves the unconditional stability of the finite difference scheme defined by Eqs. (23), (24), (26) and (29). Note that the velocities can be readily eliminated from the formulation by using the vector equations (26) and (29). Standard finite element techniques can be applied to discretize the problem leading to a set of non-linear algebraic equations for the displacements at times \( t_n \) and \( t_{n+1} \).

4. Numerical examples

In this section two numerical examples are presented to assess the advantages and drawbacks of the aforementioned energy-decaying integration scheme. The first example deals with a 2.4-m long uniform straight beam, hinged at the root so as to allow rotation about the \( \hat{e}_z \) axis and free at the tip. The physical properties of the beam are summarized in Table 1. The applied loading consists of a triangular pulse tip load, starting at \( t = 0 \), peaking at \( t = 0.025 \) and terminating at \( t = 0.05 \) s, with 1000 N peak components in both the \( \hat{e}_x \) and \( \hat{e}_y \) directions. Initially, the system is at rest with \( \mathcal{S}_0 \) aligned with \( \mathcal{F} \) at all spanwise locations.

The beam is modeled with four cubic beam elements, employing a three-point Gaussian quadrature. In order to perform a convergence study, the dynamic response of the beam is calculated with 250, 500, 1000, 2000, 4000, 8000, 16 000 and 32 000 equal time steps over the period \( t = 0 \) to \( t = 0.25 \) s. At the end of this period, the total rotation of the tip of the beam is 104.2°, which means that the effects of both large rotations and significant elastic motion are studied. In each case the dynamic response of the beam is calculated with the generalized-\( \alpha \) method [5] (\( \rho_s = 0.5 \)), and the energy-preserving and energy-decaying schemes. No analytical solution exists for this problem. The solution computed by each method with 32 000 time steps is therefore considered a ‘reference’ solution. At the final time, the reference solutions for the generalized-\( \alpha \) method, and the energy-preserving scheme predict values 9.4 \times 10^{-5} \text{ and } 1.7 \times 10^{-6} \text{ percent smaller than the energy-decaying scheme reference solution, respectively. The errors in tip displacement, velocity, and root internal moment vectors normalized with respect to the reference solution, are plotted in Figs. 6 and 7 as functions of the number of time steps. The generalized-\( \alpha \) and the energy-preserving schemes exhibit second-order accuracy, whereas the energy-decaying scheme presents third-order accuracy for the smallest time steps.}

Some of the dynamic response results computed using the energy-preserving and energy-decaying schemes with 250 and 1000 equal time steps are presented in three pairs of graphs, Figs. 8–13. The first

<table>
<thead>
<tr>
<th>Table 1 Rotating and hinged beam: physical properties</th>
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<tbody>
<tr>
<td><strong>Sectional stiffness properties</strong></td>
</tr>
<tr>
<td>Axial stiffness : 4.35080E + 07 N</td>
</tr>
<tr>
<td>Shear stiffness in ( \hat{e}_x ) direction : 1.40385E + 07 N</td>
</tr>
<tr>
<td>Shear stiffness in ( \hat{e}_y ) direction : 2.80769E + 06 N</td>
</tr>
<tr>
<td>Torsional stiffness : 2.80514E + 04 N m²</td>
</tr>
<tr>
<td>Bending stiffness, about ( \hat{e}_z ) : 2.32577E + 04 N m²</td>
</tr>
<tr>
<td>Bending stiffness, about ( \hat{e}_y ) : 2.98731E + 05 N m²</td>
</tr>
<tr>
<td><strong>Sectional mass properties</strong></td>
</tr>
<tr>
<td>Mass : 1.60920E + 00 kg/m</td>
</tr>
<tr>
<td>Mass moment of inertia about ( \hat{e}_x ) : 1.19092E + 02 kg m²</td>
</tr>
<tr>
<td>Mass moment of inertia about ( \hat{e}_y ) : 8.60200E - 04 kg m</td>
</tr>
<tr>
<td>Mass moment of inertia about ( \hat{e}_z ) : 1.10490E + 02 kg m</td>
</tr>
</tbody>
</table>
Fig. 6. Hinged beam: Error in tip displacement vector as a function of the number of steps.

Fig. 7. Hinged beam: Error in tip velocity and root internal moment vectors as a function of the number of steps.

Fig. 8. Hinged beam: Time histories of root transverse shear force $f_y$, measured in $\mathcal{S}^*$ and calculated with 250 equal time steps.

Fig. 9. Hinged beam: Spectra of root transverse shear force $f_y$, measured in $\mathcal{S}^*$.

Graph in each pair shows response time histories calculated with the energy-preserving and energy-decaying schemes with 250 steps. The second graph shows the frequency content of the responses, obtained by Fourier analysis of the time histories for both time-step sizes, between $t = 0.122$ and $t = 0.25$ s. In the case of the 250 step analysis, the Fourier analysis was performed with a 128 point FFT, and in the case of the 1000 step analysis, a 512 point FFT was used. Each frequency content graph is scaled in such a way that the vertical axis may directly be interpreted as an amplitude axis. No windowing was used during the Fourier analyses, which may lead to inaccuracies in amplitude values.
Fig. 10. Hinged beam: time histories of root torsional moment $m_1$, measured in $\mathcal{S}^*$ and calculated with 250 equal time steps.

Fig. 11. Hinged beam: spectra of root torsional moment $m_1$, measured in $\mathcal{S}^*$.

due to smearing. The frequency resolution in each case is 7.8125 Hz. It is also important to note that for the 250 step case, a frequency of 10 Hz corresponds to 100 time steps per period ($\Delta t/T = 0.01$) whereas 100 Hz corresponds to $\Delta t/T = 0.1$. At this latter frequency, the energy-decaying method already involves a significant amount of numerical dissipation (see Fig. 4). On the other hand, for the 1000 step case, a frequency of 40 Hz corresponds to $\Delta t/T = 0.01$ whereas 400 Hz corresponds to $\Delta t/T = 0.1$.

The three time histories which are plotted are the transverse shear force $f_1$ (Fig. 8) and the torsional moment $m_1$ (Fig. 10), both measured in $\mathcal{S}^*$ at the Gauss point closest to the root, and the tip angular velocity component $\omega_1$, measured in $\mathcal{S}$ (Fig. 12). All three graphs clearly show the effect of the
numerical dissipation of the energy-decaying scheme, in that the responses of this scheme clearly contain much less high frequency oscillation during the last 0.1 s than the responses of the energy-preserving scheme, and in the decaying oscillations, visible right after the termination of the applied load pulse at 0.05 s.

The four spectra of the transverse shear force $f_z$ are in reasonable agreement, considering possible errors due to smearing, with respect to the peak at 39.06 Hz. The energy-decaying scheme with 250 steps does not show any other peaks, while the corresponding energy-preserving scheme result shows peaks at 210.9, 351.6 and 414.1 Hz, correspond to $\Delta t / T = 0.21, 0.35$ and 0.41, respectively. Since no peaks at the latter two of these frequencies are noticed in the spectra of either the energy-preserving or energy-decaying schemes with 1000 steps, these may be the result of a grossly inaccurate calculation on the part of the energy-preserving scheme at high frequency.

The spectra of the torsional moments all show a prominent peak near 150 Hz. With 250 steps, the energy-preserving scheme peak is at a lower frequency than calculated with 1000 steps, showing the effect of the period elongation of this scheme when $\Delta t / T = 0.15$. With 250 steps, the energy-decaying scheme peak is quite small (0.052 Nm) compared to the energy-preserving scheme peak (1.10 Nm), showing clearly the effect of the numerical dissipation inherent to this scheme. With 1000 steps, the two methods are in good agreement with respect to the placement of the frequency of this peak, while some numerical dissipation is still evident in the energy-decaying scheme result ($\Delta t / T = 0.04$).

The spectra of the tip angular velocity show peaks at all the frequencies at which peaks are noticed in the spectra of both the transverse shear force and the torsional moment, and in general, the observations made above also apply to these spectra.

The second numerical example deals with a flexible elbow mechanism, depicted in Fig. 14. It consists of two straight 0.72-m long aluminum beams of rectangular cross-section (5 mm x 1 mm), the first of which is initially along $r_1$ and supported at its root by a hinge so as to allow rotation about the $l_3$ axis. The second beam, initially along $r_2$, is rigidly connected to the tip of beam 1. Two masses of 500 g each are rigidly connected at the tip of each of the two beams. The beam cross-sections are oriented in such a way that the smaller of the two bending stiffnesses is about the $e_3$ axis, in both cases. The modulus of elasticity, Poisson constant and density of aluminum are 73 GN/m², 0.3 and 2700 kg/m³, respectively.

The system is initially at rest. The loading of the system consists of a triangular pulse load applied to the tip of beam 2, acting in the $i_2$ direction, and a triangular pulse moment about the $l_3$ axis applied to the root of beam 1. The duration of both pulses is 5 s, peaking at 2.5 s with a 0.1 N value for the tip load and a -0.1 Nm value for the moment.

The system was modeled with 12 cubic beam elements, employing three point Gaussian quadrature, for each beam, and two rigid masses, for a total of 433 degrees of freedom. The dynamic response of the system was computed with the generalized-$\alpha$ method ($\rho_w = 0.5$), and the energy-preserving and
energy-decaying schemes, for a period of 20 s, using a total of 20,000 equal time steps. The energy-preserving scheme failed to converge at the 16,861th time step, at 16.861 s, most probably due to significant high frequency content of the predicted response.

The motion and deformation of the two beams as calculated with the energy-decaying scheme are shown in Fig. 15. In this figure, the positions of the two beams at 2-s intervals are plotted on a three-dimensional graph. The dashed lines correspond to the loci of the tips of the two beams. This graph clearly shows the extent to which elastic deformation of the beams takes place while the flexible body as a whole completes more than a full revolution about the hinge axis.

A comparison of the response time histories as calculated by the three methods is given in Figs. 16–18. Fig. 16 shows the tip displacement components of both beams in the \( \hat{i}_1 \) and \( \hat{i}_2 \) directions, while Fig. 17 shows the tip displacement components of both beams in the \( \hat{i}_3 \) direction. With respect to these two graphs it is clear that the three methods are in excellent agreement. These two graphs also clearly
show the effect of the elasticity of the system: without any elasticity the tips of the two beams would have had the same displacements in the \( \hat{t}_1 \) and \( \hat{t}_2 \) directions, while the \( \hat{t}_3 \) direction displacement component would have been zero. Fig. 18 shows the tip displacement component of beam 1, relative to the root of this beam and measured in the triad attached at this root. This graph clearly shows the bending of beam 1 under the applied moment at its root, causing the tip to have a positive relative displacement of almost 0.5 m at the peak of the applied moment, while at the same time the absolute displacement of the tip is actually small but negative (see Fig. 16). This 'elastic' displacement of 0.5 m is very large considering the 0.72-m length of the beam. The three methods are once again in good agreement, small differences being visible only with respect to the high frequency vibration between \( t = 15 \) and \( t = 20 \) s. Also, these differences are primarily between the energy-decaying scheme and the generalized-\( \alpha \) method, since the energy-preserving scheme run terminated at \( t = 16.861 \) s.

While the absolute displacement graphs are all fairly smooth, high frequency oscillations are clearly visible on the time histories of relative displacements, velocities and internal forces and moments. With respect to these higher frequency oscillations one would expect the three methods to give different
results, due to their different numerical dissipation properties. Two of these time histories together with spectra indicating their frequency content, are shown in Figs. 19–22. In both cases the spectra were obtained by Fourier analysis of the time histories between $t = 12$ and $t = 14.048$ s. Once again the frequency content graphs are scaled in such a way that the vertical axis may directly be interpreted as an amplitude axis. No windowing was used during the Fourier analyses. The frequency resolution in both cases is 0.4883 Hz. Once again, a frequency of 10 Hz corresponds to $\Delta t / T = 0.01$ whereas 100 Hz corresponds to $\Delta t / T = 0.1$.

Fig. 19 shows the time history of the transverse shear force in the $\hat{e}_3$ direction at the Gauss point closest to the root of beam 1, as calculated by the energy-decaying scheme. Fig. 20 compares the spectrum of this time history with those of the other two methods. The three methods are in good agreement with respect to peaks at 14.65 and 25.39 Hz. In the region 40 to 55 Hz peaks predicted by the energy-decaying scheme are substantially smaller than those corresponding to the energy-preserving scheme, and to a lesser extent, the generalized-$\alpha$ method. At 50 Hz ($\Delta t / T = 0.05$) the system has completed 600 oscillations from $t = 0$ to the point in time where the Fourier analysis starts, allowing enough time for the oscillation to decay, even with a small amount of algorithmic damping. On the other hand, all three methods are in good agreement with respect to the peak value near 60 Hz, an observation which cannot be explained.

Fig. 21 shows the time history of the bending moment about the $\hat{e}_2$ axis at the Gauss point closest to the root of beam 2, as calculated by the energy-decaying scheme. Fig. 22 compares the spectrum of this time history with those of the other two methods. The three methods are in good agreement with respect to peaks at 5.37, 14.65 and 24.90 Hz. With respect to peaks near 50 Hz, the energy-decaying scheme once again predicts substantially lower levels than the other two methods, but at about 60 Hz the three methods are in good agreement.

The dynamic response calculation was repeated with the energy-decaying scheme with 10,000 steps over the 20-s period. The normalized relative energy is calculated as $(E(t) - E_r) / E_r$, where the reference energy level $E_r$ is the total energy in the system at the termination of the applied loading pulses. The time history of the normalized relative energy with 20,000 and 10,000 steps, is plotted in Fig. 23. A larger time step clearly causes larger decay in energy. Fig. 24 shows the time history of the increment in energy $\Delta E = E_n - E_{n-10}$, in the case of the 20,000 step analysis, and $\Delta E = E_n - E_{n-5}$, in the case of the 10,000 step analysis, where the subscript $n$ indicates the $n$th time step. The increment in energy was therefore calculated over the same time increments of 0.01 s in both cases. As expected, the analysis with the larger time step results in substantially higher, in absolute value, increments in energy.
5. Discussion and conclusions

In this paper, an unconditionally stable scheme was presented for the time integration of the non-linear equations of motion of beams. The proposed scheme is of a finite difference nature, though it mimics the scheme obtained by applying a time discontinuous Galerkin method to a single degree of freedom linear oscillator. An energy-decay inequality obtained by a direct computation of the work done by the discretized inertial and elastic forces over a time step is shown to be implied.

The scheme presents the following features. First, the energy-decay inequality provides a rigorous proof of the unconditional stability of the scheme. Second, numerical experimentation indicates that the salient features of the underlying time discontinuous Galerkin method are inherited by the proposed scheme. In particular, high frequency numerical dissipation with asymptotic annihilation is achieved. Finally, the non-linear equations of motion need not be recast in the symmetric hyperbolic form, which does not seem possible in this case.

Though more expensive than conventional schemes in structural dynamics, the proposed energy-decaying scheme seems more effective than time discontinuous Galerkin methods. At each time step two sets of displacements are computed, one across the jump, and one at the end of the time step. Velocities are readily eliminated from the formulation. There is no need to numerically perform time integrals over the time step.

Though this paper focused on the non-linear equations of motion of beams, the proposed approach can be extended to treat two- and three-dimensional elastodynamics, as well as multi-body dynamic problems.

The total energy loss at each time step could be used as a time step control parameter. Indeed, the total energy loss is a measure of the global accuracy of the time integration process. If this total energy loss is larger than a preset value, a smaller time step would be used.

Acknowledgement

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Appendix A. The conformal rotation vector

Let \( \epsilon \) and \( e \) be the Euler parameters representing a finite rotation [15]. Consider now the following conformal transformation which defines \( \epsilon_0 \) and the components of the conformal rotation vector \( c \):

\[
\begin{align*}
\epsilon_0 &= \frac{4\epsilon_0}{1 + \epsilon_0}; \quad e = \frac{4\epsilon}{1 + \epsilon_0}; \quad \epsilon_0 = \frac{c_0}{4 - c_0}; \quad e = \frac{c}{4 - c_0},
\end{align*}
\]

(A.1)

where the dependent parameter \( \epsilon_0 = 2 - (c_1^2 + c_2^2 + c_3^2)/8 \).

The geometric interpretation of the conformal rotation vector is easily derived from its definition as

\[
e = 4u \tan \frac{\phi}{4},
\]

(A.2)

where \( \phi \) is the magnitude of the finite rotation and \( u \) the components of the unit vector about which it takes place. The following matrix is defined:

\[
G(c) = \frac{1}{4 - c_0} \left( c_0 I + \frac{c \cdot c^T}{4} \right).
\]

(A.3)

It enjoys the following remarkable properties:

\[
GG^T = I; \quad Gc = c,
\]

(A.4)

\[
G + \frac{\hat{c}^T}{4 - c_0} = G^T - \frac{\hat{c}^T}{4 - c_0} = \left( \frac{G + G^T}{2} \right)
\]

(A.5)

\[
\left( I + \frac{\hat{c}^T}{c_0} \right) G = \left( I - \frac{\hat{c}^T}{c_0} \right) G^T = \left( \frac{G + G^T}{2} \right).
\]

(A.6)

\[
\frac{2\hat{c}}{4 - c_0} = G - G^T.
\]

(A.7)

The rotation matrix defined by \( c \) easily writes as

\[
R(c) = \frac{1}{(4 - c_0)^2} \left( c_0 I + 2c_0 \hat{c} + \hat{c} \hat{c} + c \cdot c^T \right) = G(c)G(c).
\]

(A.8)

This last relationship shows that the conformal rotation vector can be conveniently used to express the half rotation (through \( G(c) \)), and the rotation (through \( R(c) \)). All these expressions are purely algebraic.

Appendix B. Discretization of the finite rotations

Consider an initial time \( t_i \), a final time \( t_f \) and a mid-point time \( t_m = (t_i + t_f)/2 \) and the corresponding triads \( \mathcal{T}_i, \mathcal{T}_f \) and \( \mathcal{T}_m \), respectively. The rotation matrices associated with those triads are \( R_i, R_f \) and \( R_m \), respectively, all measured in \( \mathcal{T} \). \( R(c) \) is the rotation matrix from \( \mathcal{T}_i \) to \( \mathcal{T}_f \), measured in \( \mathcal{T} \). The mid-point triad \( \mathcal{T}_m \) is defined so that the rotation from \( \mathcal{T}_i \) to \( \mathcal{T}_m \) and \( \mathcal{T}_m \) to \( \mathcal{T}_f \) are equal. Let \( G \) be that rotation, measured in \( \mathcal{T} \). Let \( G^* \) and \( R^* \) be the corresponding rotation matrices measured in \( \mathcal{T}_h \). The following relationships are readily derived:

\[
\begin{align*}
G &= R_m R_i^T; \quad G^* = R_o^T R_i^T R_m R_o; \\
G &= R_f R_i^T; \quad G^* = R_o^T R_h^T R_i R_o; \\
R &= R_i R_f^T; \quad R^* = R_o^T R_m R_i R_o^T R_m R_f R_o^T R_m R_f R_o^T R_m R_f R_o^T R_m R_f R_o.
\end{align*}
\]

(B.1)

and
Finally, the following notations are introduced:

\[
\mathcal{R}_i = \begin{bmatrix} R_i R_0 & 0 \\ 0 & R_i R_0 \end{bmatrix}, \quad \mathcal{R}_t = \begin{bmatrix} K_t R_0 & 0 \\ 0 & K_t R_0 \end{bmatrix}. \tag{B.3}
\]

**Appendix C. Discretization of the equations of motion of a beam**

Consider the following discretization of the beam equations of motion (2.2):

\[
\begin{align*}
\frac{\mathcal{R}_i p^*_i - \mathcal{R}_t p^*_t}{\Delta t} + \mathcal{U} \left[ \frac{u_i - \bar{u}_i}{\Delta t} \right] p^*_i + p^*_t &- (\mathcal{B}_h g^*_h)' - \mathcal{U} \left[ \frac{2}{c_0} (\bar{u}_h^* + \bar{u}_h^*) \right] \mathcal{B}_h g^*_h = - q_h. \tag{C.1}
\end{align*}
\]

where

\[
\mathcal{B}_h = \begin{bmatrix}
2 R_i R_0 + R_i R_0 & 0 \\
2 R_i R_0 & 4 - c_0
\end{bmatrix},
\]

the midpoint displacement is defined as \( u_h = (u_i + u_t)/2 \), and the elastic forces \( g^*_h \) will be determined later.

Premultiplying these discretized equations by

\[
\left[ \frac{u^*_i - u^*_t}{\Delta t} c^T \right]
\]

and integrating over the span of the beam yields

\[
\int_0^L \left[ \frac{u^*_i - u^*_t}{\Delta t} c^T \right] \left[ \frac{\mathcal{R}_i p^*_i - \mathcal{R}_t p^*_t}{\Delta t} \right] + \mathcal{U} \left[ \frac{u_i - \bar{u}_i}{\Delta t} \right] \left[ \frac{2 R_i R_0}{4 - c_0} \right] p^*_i + p^*_t \]

\[
- (\mathcal{B}_h g^*_h)' - \mathcal{U} \left[ \frac{2}{c_0} (\bar{u}_h^* + \bar{u}_h^*) \right] \mathcal{B}_h g^*_h
\]

\[
\Delta W = \int_0^L \mathcal{B}_h g^*_h \ dx_1 = 0.
\]

Integration by parts now yields

\[
\int_0^L \left[ \frac{u^*_i - u^*_t}{\Delta t} c^T \right] \left[ \frac{\mathcal{R}_i p^*_i - \mathcal{R}_t p^*_t}{\Delta t} \right] + \mathcal{U} \left[ \frac{u_i - \bar{u}_i}{\Delta t} \right] \left[ \frac{2 R_i R_0}{4 - c_0} \right] p^*_i + p^*_t \]

\[
+ \int_0^L \left[ \left[ \frac{u^*_i - u^*_t}{\Delta t} c^T \right] - \left[ \frac{u^*_i - u^*_t}{\Delta t} c^T \right] \mathcal{U} \left[ \frac{2}{c_0} (\bar{u}_h^* + \bar{u}_h^*) \right] \mathcal{B}_h g^*_h \ dx_1 = \Delta W \Delta t. \tag{C.2}
\]

where \( \Delta W \) is the work done by the externally applied forces during a time step.

Focussing on the inertial terms in the above equation and using relationships (B.2), the first integral writes

\[
\int_0^L \left[ \frac{u^*_i - u^*_t}{\Delta t} R_h R_0 c^{*T} \right] \left[ \begin{bmatrix} G^* + \frac{\bar{c}^{*T} T}{4 - c_0} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \frac{\mathcal{U}^*_i - \mathcal{U}^*_t}{\Delta t} \right] - \left[ \begin{bmatrix} G^* + \frac{\bar{c}^{*T} T}{4 - c_0} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \frac{\mathcal{U}^*_i - \mathcal{U}^*_t}{\Delta t} \right] \ dx_1.
\]

Invoking first Eq. (A.5) and then once more relationship (B.2), this integral may be written as

\[
\int_0^L \mathcal{V}_h^* \frac{\mathcal{U}^*_i - \mathcal{U}^*_t}{\Delta t} \ dx_1 = \frac{1}{\Delta t} \int_0^L \mathcal{V}_h^* M^* (\mathcal{V}^*_i - \mathcal{V}^*_t) \ dx_1, \tag{C.3}
\]
where
\[ \mathbf{v}_h^{*T} = \left[ \frac{u_T^* - u_T^*}{\Delta t} R_i R_o + R_i R_o \frac{e^{*T}}{2} \right]. \]

Focussing now on the elastic terms in Eq. (C.2), the second integral writes
\[ \frac{1}{\Delta t} \int_0^L g_{h}^{*T} \left\{ \partial_h \left[ u'_h - u'_h \right] \right\} - \partial_h^T \left[ \frac{2}{c_0} (\bar{u}_h^* + \bar{u}_h^*) \right] \right\} \, dx. \]  
(C.4)

Using relationships (B.2), the first three components of the vector between the braces in Eq. (C.4) can be written as
\[ \frac{G^* + G^*}{2} R_0^T R_h \left[ (u_0' + u_0') - (u_0' + u_0') + R_0 R_0 \frac{c^*}{c_0} R_0^T R_h (u_0' + u_0') + u_0' + u_0' + u_0' \right]. \]

With the use of, once again, relationships (B2), and Eq. (A.6), the above expression can be further simplified to
\[ R_0^T R_h (u_0' + u_0') - R_0^T R_h (u_0' + u_0') = e^*_i - e^*_i. \]  
(C.5)

The last three components of the vector between the braces in Eq. (C.4) write:
\[ \frac{2R_0^T R_h}{4 - c_0^*} (R_h R_o e^*)'. \]

With the help of relationships (B.2) it may be shown that
\[ \frac{2R_0^T R_h}{4 - c_0^*} (R_h R_o e^*)' = \frac{2G^*}{4 - c_0^*} \left[ e^{*T} \left( c^* + (k_0^* + k_1^*) e^* \right) \right] = k_1^* - k_1^*. \]  
(C.6)

Combining results (C.5) and (C.6), the elastic terms write
\[ \frac{1}{\Delta t} \int_0^L g_{h}^{*T} (e_i^* - e_i^*) \, dx. \]  
(C.7)

Defining the elastic forces \( g_{h} = C^* e_h^* \), where \( e_h^* \) are yet to be defined sectional strains, the expression (C.7) may be written as
\[ \frac{1}{\Delta t} \int_0^L e_i^{*T} C^* (e_i^* - e_i^*) \, dx. \]  
(C.8)

Substituting the final expressions from (C.3) and (C.8) into Eq. (C.2) results in
\[ \int_0^L v_i^{*T} M^* (v_i^* - v_i^*) + e_i^{*T} C^* (e_i^* - e_i^*) \, dx = \Delta W. \]  
(C.9)

This fundamental scalar relationship is implied by the discretization (C.1) of the beam equations of motion.

References