Adjoint method for the sensitivity analysis of composite beam cross-sections*

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Abstract

The structural components found in mechanical and aerospace engineering applications are often idealized as beams. The sectional stiffness and mass properties of beams made of homogeneous materials and presenting simple geometric shapes are evaluated easily. Axial and bending stiffness constants can be evaluated through simple formulas, whereas the computation of torsional stiffness is more cumbersome. In many aerospace and mechanical applications, wings and rotor blades present complex geometric shapes and are made of advanced, highly anisotropic composite materials. In such cases, the evaluation of the sectional stiffness properties is a complex task that requires the use of finite element models of the cross-section. The gradient-based optimization of these complex systems requires the evaluation of design sensitivities, which, in turn, calls for the evaluation of the sensitivity of sectional properties with respect to the parameters that define the configuration of the section. This paper presents an approach to this problem based on the adjoint method. Adjoint equations that enable the efficient evaluation of sensitivity derivatives of sectional stiffness properties with respect to composite design parameters are derived. Examples of cross-sections presenting various configurations and made of advanced composite materials are investigated. Real- and complex-step numerical differentiation methods are used to verify the proposed adjoint formulation.

1 Introduction

When simulating the dynamic behavior of flexible multibody systems, structural components are often approximated as beams. Geometrically exact beams [1] have become the workhorse of a number of codes for the comprehensive analysis of flexible multibody systems [2–4]. A finite element based tool that evaluates the dynamic response of geometrically exact beams, Dymore, has been developed in recent years. A detailed description of this code is beyond the scope of this paper; its

Theoretical formulation is established in the textbook by Bauchau [3], and example applications are described in numerous publications therein and in the review article by Bauchau et al. [4, 5]. The geometrically exact beam formulation deals with beams undergoing arbitrarily large rotation and translation but subjected to small deformation; the formulation satisfies the principle of objectivity, i.e., it remains invariant under the superposition of large rigid-body motion.

The success of this approach hinges upon the accurate evaluation of the sectional stiffness characteristics of the beam: indeed, this data is an input to the beam model. For beams made of homogeneous materials and presenting simple geometric shapes, simple formulas to evaluate axial and bending stiffness constants are available in textbooks [6]. Typically, the computation of torsional stiffness is more cumbersome: for instance, the torsional stiffness of a rectangular section is expressed as an infinite series [6, 7]. For most sectional shapes, closed-form solutions are not available for the torsional and shear stiffnesses.

When dealing with more complex shapes and materials, the evaluation of the sectional stiffness coefficients becomes more involved. Rather than speaking of “sectional stiffness coefficients,” it becomes necessary to introduce the concept of “sectional stiffness matrix.” This matrix of size $6 \times 6$, denoted $K$, relates the sectional stress resultants, $\mathcal{F}$, to the sectional deformation measures, $\mathcal{E}$, so that $\mathcal{F} = K \mathcal{E}$.

In mechanical engineering applications, high-precision mechanisms present complex geometries and rely on advanced materials. In such cases, the sectional stiffness matrix becomes fully populated and its evaluation becomes a complex task that requires the use of finite element models of the cross-section. In aerospace applications, wings or rotor blades also present complex geometric shapes and are made of advanced, highly anisotropic composite materials.

A cursory look at the available literature shows that the analysis of such complex cross-sections is no simple matter. The sectional stiffness matrix is a byproduct of the solution to Saint-Venant’s problem, which provides the theoretical foundations for beam theory. For straight beams made of heterogenous materials, Iesan [8, 9] developed a systematic approach to solve Saint-Venant’s problem. Dong et al. [10, 12] generalized Iesan’s method using a finite element discretization for the cross-section. Berdichevsky [13] proposed the Variational Asymptotic Method (VAM), in which asymptotic analysis is applied to the energy functional. In this approach, Saint-Venant’s problem is reduced to a two-dimensional analysis over the beam’s cross-section. A unified beam theory based on VAM was further refined by Atilgan et al. [14, 15], Hodges [16], and Yu et al. [17].

Giavotto et al. [18] presented a comprehensive solution strategy for Saint-Venant’s problem. Their approach is based on a two-dimensional analysis of the beam’s cross-section using finite elements and provides its stiffness characteristics in the form of a $6 \times 6$ sectional stiffness matrix. Furthermore, the three-dimensional strain field at any point of the cross-section can be recovered once the sectional strains are known. Their work also identifies the two types of solutions present in beams: the central solutions and the extremity solutions, as should be expected from Saint-Venant’s principle. Borri et al. [19] generalized this methodology to naturally curved beams; the magnitudes of beam’s initial curvatures are not required to be small in their approach.

Bauchau and Han [20] have formulated beam problems using Hamilton’s canonical formulation and have shown that exact solutions of the three-dimensional theory of elasticity can be derived for beams of general cross-sectional shape made of anisotropic materials. The same authors generalized the formulation to initially curved and twisted beams undergoing arbitrary large motion, proving that the governing equations for geometrically exact beams can be rigorously derived from three-dimensional elasticity [21]. The three-dimensional stress field is a byproduct of these exact
solutions. Based on this formulation, a finite-element code that evaluates the exact solutions of three-dimensional elasticity for beams with general cross-sections, SectionBuilder, was developed.

Figure 1 illustrates the strategy for the analysis of flexible multibody systems: the problem is decomposed into a linear, two-dimensional analysis of the cross-section (the “Sectional Analysis”) and a nonlinear, one-dimensional analysis of geometrically exact beams (the “Beam Analysis”). The data flow between these two analysis modules is also indicated. Realistic systems will involve several beam components and each beam could involve many different cross-sections along its span. Based on the geometry of the cross-section, material properties, and fiber orientation, the sectional analysis code computes its properties. In turn, these sectional characteristics are inputs to the beam analysis that computes the dynamic response of the system. Once the beam analysis is completed, the time history of the sectional loads is computed and the recovery relationships then yield the three-dimensional stress and strain fields at any point of the beam element. This decomposition strategy enables a very accurate analysis of complex systems at a very modest computational cost. In the context of this work, SectionBuilder and Dymore are used as the sectional and beam analysis tools, respectively.

Due to its multidisciplinary nature, the design and optimization of flexible multibody systems is a complex task. A number of approaches presenting different levels of fidelity and performance have been developed. Recent surveys of the existing methodologies can be found in Blasques and Stolpe [22] and Tromme et al. [23], who categorize the design strategies in terms of the class of structural optimization, the modeling accuracy associated with the flexible multibody formulation, and the level of coupling between the static and dynamic analyses.

In the present approach, the design and optimization tasks reflect the two steps shown in fig. 1: the first step operates on the sectional analysis tool whereas the second operates on the beam analysis tool. The two steps are coupled through an iterative process. This paper focuses on the first step of the process, the design and optimization of cross-sections.

The sectional or local design and optimization procedure operates on the linear, two-dimensional finite element analysis of a single cross-section located at a specific span-wise position along the beam, as depicted in fig. 2. This problem involves a large number of design parameters: material selection, ply thickness and fiber orientation, and geometry and topology of the section. The objective of the optimization procedure is to achieve target values of sectional stiffness and mass matrices while satisfying constraints on local three-dimensional stress levels.

Note that sectional parameters, such as bending and torsional stiffness or mass per unit span, are not appropriate design variables, especially when composite materials are present. While the
configuration of a cross-section defines the sectional stiffness and mass matrices uniquely, the reverse, however, is not true. Indeed, an infinite number of cross-sectional configuration can present the same sectional stiffness and mass matrices and it is also possible that no cross-sectional configuration exists that will achieve the desired sectional properties. Clearly, the sectional analysis tool establishes the link between the physical design parameters and the sectional stiffness and mass properties of the cross-section.

Furthermore, for many systems, manufacturing the beam is a complex task. Consider, for instance the manufacturing of a wind turbine blade. Metal designs are too heavy, pointing towards the use of glass fibers. Glass fibers, however, cannot achieve the required stiffness and hence, a mixture of glass and carbon fibers is often selected. Given the size of the blade, the manufacturing process imposes technological constraints on most design parameters that must be taken into account in the optimization process. Clearly, such design and optimization process is possible only when expressed in terms of design variables such as material selection, ply thickness and fiber orientation, and geometry and topology of the section: the sectional analysis is an indispensable part of the design process.

Figure 2: The sectional optimization problem

Figure 3: The beam optimization problem

For beams presenting simple cross-sectional configurations and made of homogeneous isotropic materials, simpler design and optimization strategies have been used [24–26]; Tromme et al. [27] provide a comprehensive review of the topic. The in-depth approach illustrated in fig. 1 has been used primarily by researchers dealing with aerospace [22, 28–32] and wind turbine [33] problems. Other uses of design and sensitivity analysis within the analysis of composite cross-sections include the validation of homogenization methods [34].

The beam or global design and optimization procedure operates on the nonlinear, one-dimensional finite element analysis of geometrically exact beams, as depicted in fig. 3. In addition to beams, the multibody system could include a number of other components such as rigid-body, plate, or shell elements, as well as force elements such as springs, dampers, and actuators. A key set of design parameters are the sectional stiffness and mass characteristics at the various span-wise locations along the blade. These sectional properties encapsulate the intricate details of the various cross-sectional constructions into a small number of physically meaningful parameters.

When using gradient-based optimization procedures, the derivatives of the objective function with respect to the design parameters, often called “design sensitivities,” must be evaluated. As the number of design variables grows, the evaluation of design sensitivities becomes increasingly burdensome and the use of finite differences becomes prohibitive [35,36]. These requirements call for an efficient sensitivity analysis that covers all stages of the design process.
The adjoint variable method [37–39] and the direct differentiation method [40,41] are the most widely used methods for the efficient, analytical computation of design sensitivities. The computational cost of the adjoint variable method is nearly independent of the number of design parameters, whereas the cost of the direct differentiation method is nearly independent of the number of objective functions and optimization constraints. The multidisciplinary nature of aerospace problems typically results in a large number of design variables, particularly when dealing with computational fluid dynamics methods and with design parameters related to the mesh geometry [36, 42–44]. In those fields, the adjoint method is often the approach of choice and it seems natural to use the same approach for the structural problem. The basic idea behind the adjoint method is that the derivative of the state variables with respect to the design parameters is eliminated through the judicious choice of the adjoint variables.

Within the field of flexible multibody dynamics, some authors have used the adjoint method [32, 45–47], whereas others prefer direct methods [24,26,27,31,34,48–51], to deal with design optimization, parameter identification, control, or uncertainty quantification. Alternative differentiation tools have also been used, including complex-valued finite differences [34, 52, 53] and automatic differentiation [29, 54]. These are often used for the verification of other methods or to bypass the analytical differentiation of complicated model subcomponents. Finally, some approaches avoid the evaluation of sensitivity derivatives: they implement finite-difference methods [22,25,33], design of experiments (DOE) methodologies [28,30], or genetic algorithms [55].

This paper presents the development and implementation of the adjoint method for the evaluation of sensitivity derivatives of beam sectional properties. Particular emphasis is placed on beams made of anisotropic composite materials. Such analysis is rarely used in flexible multibody dynamics applications, which often rely on simplified beam formulations and simple cross-sectional shapes for which analytical expressions of the sectional stiffness constants are available. The two-dimensional finite element approach to the evaluation of the sectional properties is not a standard approach and, furthermore, the splitting of the problem into the sectional and beam analyses to perform optimization is unique. The main goal of the paper is the efficient evaluation of the sensitivity derivatives of the sectional stiffness matrix. This is the first step towards the efficient evaluation of sensitivity derivatives for flexible multibody dynamics problems.

2 Governing equations

This section presents a summary of SectionBuilder’s governing equations; further details can be found in Bauchau and Han [20, 21, 56]. The approach is based on the semi-discretization of the beam, in which the finite-element mesh extends only over the cross-section, as shown in fig. 4. The discretization involves a total of $m$ nodes. The displacement of an arbitrary point in the cross-section is expressed as

$$u(\alpha_1, \alpha_2, \alpha_3) = \mathbf{N} (\alpha_2, \alpha_3) \hat{\mathbf{u}} (\alpha_1),$$

where matrix $\mathbf{N}$, of size $3 \times n$, stores the shape functions, array $\hat{\mathbf{u}}$, of size $n$, stores the nodal coordinates, and $\alpha_1, \alpha_2, \alpha_3$.
and \( \alpha_3 \) are the material coordinates of a point, as shown in fig. 4. The problem presents a total of \( n = 3m \) degrees of freedom. The nodal displacements depend on axial material coordinate \( \alpha_1 \), whereas the shape functions depend on sectional coordinates \( \alpha_2 \) and \( \alpha_3 \). Bases \( I = (\bar{i}_1, \bar{i}_2, \bar{i}_3) \) and \( B = (\bar{b}_1, \bar{b}_2, \bar{b}_3) \) are the inertial basis and the material basis attached to the cross-section, respectively.

The computation of the strain energy of the system gives rise to the following stiffness matrices, all of size \( n \times n \),

\[
\begin{align*}
M & = \int_A A^T \frac{D}{d} A \sqrt{g} \, dA, \\
C & = \int_A B^T \frac{D}{d} A \sqrt{g} \, dA, \\
E & = \int_A B^T \frac{D}{d} B \sqrt{g} \, dA, 
\end{align*}
\]  

(2a) (2b) (2c)

where \( A \) is the cross-sectional area of the beam, \( \sqrt{g} \) is the Jacobian of the local coordinate transformation, and matrices \( M \) and \( E \) are symmetric. Matrices \( A_L = A N \) and \( B_L = B N \) are used to compute the three-dimensional strain components; a detailed derivation of differential operators \( A \) and \( B \), of size 6 \( \times \) 3, is found in Han and Bauchau [21]. Matrix \( D \), of size 6 \( \times \) 6, contains the three-dimensional material stiffness constants, i.e., it relates the six stress components stored in \( \tau_c \) to the six strain components stored in \( \gamma_c \), so that \( \tau_c = D \gamma_c \). All are resolved in basis \( B \).

The three in-plane coordinates of the nodes, \( q_i^T = \{0, \alpha_2, \alpha_3\} \), are stacked in \( n \times 6 \) matrix \( Z \),

\[
\begin{align*}
\bar{z}_i & = \left[ I, q_i^T \right], \\
Z & = \left[ \bar{z}_1, \ldots, \bar{z}_m \right]^T,
\end{align*}
\]  

(3a) (3b)

where notation \( \tilde{q}_i \) indicates the skew-symmetric matrix associated with vector \( q_i \). For convenience, the 6 \( \times \) 6 Timoshenko stiffness matrix, \( C_T = Z^T M Z \), and compliance matrix, \( S_T = C_T^{-1} \), are introduced. These correspond to the sectional compliance and stiffness matrices obtained from Timoshenko’s beam theory [7].

The global equilibrium equations are expressed in terms of matrix \( W_0 \), of size \( n \times 6 \), whose six columns store the nodal warping components associated with a unit axial force, unit shear forces along unit vectors \( \bar{i}_2 \) and \( \bar{i}_3 \), a unit torque, and unit bending moments about unit vectors \( \bar{i}_2 \) and \( \bar{i}_3 \), respectively. For straight beams, the governing equations are expressed as

\[
\begin{align*}
\xi X_1 & = B_1, \\
\xi X_0 & = B_0 - C X_1.
\end{align*}
\]  

(4a) (4b)

Note the recursive nature of the equations, where the solution of linear system (4a) appears on the right-hand side of linear system (4b). The system matrices are defined as

\[
\begin{align*}
\xi & = \begin{bmatrix} E \; (C Z)^T \; (C Z) \end{bmatrix}, \\
C & = \begin{bmatrix} C - C_T \; -(M Z)^T \; \begin{bmatrix} 0 \; 0 \end{bmatrix} \end{bmatrix},
\end{align*}
\]  

(5a) (5b)
and the following notation is used,

\[
X_1 = \begin{bmatrix} W_1 \\ Q_1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} W_0 \\ Q_0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0_{n \times 6} \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0_{n \times 6} \end{bmatrix}.
\] (6)

where \( \tilde{K} \) denotes the \( 6 \times 6 \) local-frame curvature tensor. System matrix \( \varepsilon \), of size \((n + 6) \times (n + 6)\), can be shown to be six times singular and hence, the solution process yields one of infinite particular solutions \( X_0 \), which contains a particular warping field, \( W_0 \). Submatrices \( Q_0 \) and \( Q_1 \) are byproducts of the computation that will not be necessary in what follows. Submatrix \( \frac{W_0}{Q_0} \) is, therefore, the main result from the equilibrium equations.

Compliance matrix \( S = \tilde{K}^{-1} \), of size \( 6 \times 6 \), relates the six sectional stress resultants, \( F_c \), to the six sectional deformation measures, \( E_c \), which include 3 strain components and 3 curvature components, so that \( E_c = \hat{S} F_c \). The compliance matrix associated with the particular solution of system (4) is

\[
S_0 = S_T \left[ I_6 - (M Z)^T \tilde{W}_0 \tilde{K}^T - (C Z)^T \tilde{W}_0 \right].
\] (7)

The general solution \( \hat{W} \) that satisfies the overall equilibrium equations of the problem, along with the associated compliance matrix \( \hat{S} \), can then be obtained as follows,

\[
\alpha = - \left[ M \left( Z S_0 + W_0 \tilde{K}^T \right) + C^T W_0 \right]^T \tilde{W}_0,
\] (8a)

\[
\hat{W} = \tilde{W}_0 + Z \alpha,
\] (8b)

\[
\hat{S} = S_0 - \tilde{K} \alpha - \alpha \tilde{K}^T.
\] (8c)

Equation (8c) provides the compliance matrix of the beam; this matrix is a required input for the analysis of geometrically exact beam problems. The goal of this paper is to compute the sensitivity derivatives of this compliance matrix with respect to the parameters that define the configuration and physical properties of the cross-section.

### 3 Design optimization

Typically, sensitivity analysis is developed and implemented within the context of optimization problems. In the present case, the output of the sectional analysis is the compliance matrix, \( \hat{S}(\tilde{W}_0, (X_0, b)) \), where array \( b \) stores the \( n_b \) design parameters of the problem. Figure 5 illustrates some of the parameters that would be relevant for a helicopter or wind turbine blade design. These parameters could include geometric parameters such as the thickness of the inner and outer wrap, denoted by \( t_I \) and \( t_O \), respectively, or the extent of the connection between the inner and outer wrap along the upper skin, denoted by \( \eta_{IT} \) and \( \eta_{2T} \).

Additional parameters could include material properties such as the stiffness parameters of the material in the longitudinal and transverse directions, denoted \( E_1 \) and \( E_2 \), respectively, or the fiber orientation angle, \( \gamma \). All these parameters define the material stiffness matrix, \( \tilde{D} \), see eq. (2), and hence, they affect the compliance stiffness matrix. This paper focuses on the evaluation of the sensitivity derivatives with respect to material parameters.
\[ (\mathcal{D}^+)^{-1} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & -\nu_{12}/E_1 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & -\nu_{23}/E_2 & 1/E_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1 + \nu_{23})/E_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}, \tag{9} \]

where \( E_1 \) and \( E_2 \) are Young’s moduli, \( G_{12} \) is the shear modulus, and \( \nu_{12} \) and \( \nu_{23} \) are Poisson’s ratios; these five engineering constants describe the anisotropic stiffness of the material. Stiffness matrix \( \mathcal{D} \) in eqs. (2) stores the components of the material stiffness matrix resolved in sectional basis \( \mathcal{B} \).

The relationship between the stiffness components resolved in bases \( \mathcal{M}^+ \) and \( \mathcal{B} \) is

\[ \mathcal{D} = R_{\mathcal{B}} \mathcal{D}^+ R_{\mathcal{B}}^T, \tag{10} \]

where rotation matrix \( R_{\mathcal{B}} \) is defined as

\[ R_{\mathcal{B}} = \begin{bmatrix} \ell_1^2 & m_1^2 & n_1^2 & 2m_1n_1 & 2\ell_1n_1 & 2\ell_1m_1 \\ \ell_2^2 & m_2^2 & n_2^2 & 2m_2n_2 & 2\ell_2n_2 & 2\ell_2m_2 \\ \ell_3^2 & m_3^2 & n_3^2 & 2m_3n_3 & 2\ell_3n_3 & 2\ell_3m_3 \\ \ell_1\ell_3 & m_1m_3 & n_1n_3 & m_1n_3 + m_3n_1 & \ell_1n_3 + \ell_3n_1 & \ell_1m_3 + \ell_3m_1 \\ \ell_1\ell_2 & m_1m_2 & n_1n_2 & m_1n_2 + m_2n_1 & \ell_1n_2 + \ell_2n_1 & \ell_1m_2 + \ell_2m_1 \end{bmatrix}. \tag{11} \]

Vectors \( \ell, m, n \) are the columns of the rotation matrix that brings the sectional basis to the material basis

\[ R = [\ell, m, n] = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma \cos (\alpha + \beta) & \cos \gamma \cos (\alpha + \beta) & -\sin (\alpha + \beta) \\ \sin \gamma \sin (\alpha + \beta) & \cos \gamma \sin (\alpha + \beta) & \cos (\alpha + \beta) \end{bmatrix}, \tag{12} \]

where angles \( \alpha, \beta, \) and \( \gamma \) define the orientation of basis \( \mathcal{M}^+ \) with respect to basis \( \mathcal{B} \), as shown in fig. 4. Specifically, angle \( \alpha \) defines the planar rotation about unit vector \( \vec{b}_1 \) that brings unit vector \( \vec{b}_2 \) to the finite element unit vector \( \vec{u} \). Angle \( \beta \) defines the planar rotation about unit vector \( \vec{b}_1 \) that brings unit vector \( \vec{u} \) to the material frame. Finally, angle \( \gamma \) defines the fiber orientation angle. For standard lay-up configurations, \( \beta = 0 \).
4 Adjoint sensitivity analysis

The main output of sectional analysis tools such as SectionBuilder is the beam’s sectional compliance matrix, which depends on the sectional warping field, see eqs. (8). In this section, the sensitivity derivative of the sectional compliance matrix is evaluated using the adjoint method. The computational cost of this approach is similar to the cost of the evaluation the sectional compliance matrix itself, regardless of the number of design parameters [37–39]. An early version of these equations that considered a single component of the compliance matrix was presented by the authors in [57]; in this early work, far simpler numerical examples than the ones shown here were analyzed.

4.1 Vectorized governing equations

Governing equations (4) are matrix equations. To simplify the development of the sensitivity analysis, they are recast in vectorized form as

\[ L_1(X_1, b) \equiv \mathcal{E}'X_1 - \mathcal{B}_1 = 0, \]  
\[ L_0(X_0, X_1, b) \equiv \mathcal{E}'X_0 - \mathcal{B}_0 + \mathcal{C}'X_1 = 0, \]

where the matrices of unknown variables have been vectorized as \( X_1 = \text{vec}(X_1) \), \( X_0 = \text{vec}(X_0) \), \( \mathcal{B}_1 = \text{vec}(\mathcal{B}_1) \), and \( \mathcal{B}_0 = \text{vec}(\mathcal{B}_0) \). Furthermore, the system matrices have been recast as

\[ \mathcal{E}' = \mathcal{I}_6 \otimes \mathcal{E}, \]  
\[ \mathcal{C}' = \mathcal{I}_6 \otimes \mathcal{C}. \]

Matrix \( \mathcal{I}_6 \) denotes the identity matrix of size 6 \( \times \) 6 and \( \otimes \) indicates the Kronecker product.

4.2 Adjoint equations

The Lagrangian of the system is defined as

\[ \mathcal{L} = S + \Lambda_1^T L_1 + \Lambda_0^T L_0, \]

where \( S = \text{vec}(\mathcal{S}) \) is the vectorized form of the compliance matrix and matrices \( \Lambda_1 \) and \( \Lambda_0 \) are matrices of Lagrange multipliers, also called “adjoint variables,” which enforce vectorized governing equations (13). Adjoint variable matrices are used instead of vectors because the sensitivity of all the components of compliance matrix \( S \) must be evaluated. In contrast, typical adjoint methods consider only one output or objective function.

If the governing equations are satisfied, the Lagrangian reduces to the compliance matrix, \( \mathcal{L} = S \), leading to the design sensitivities,

\[ \frac{d\mathcal{L}}{db} = \frac{dS}{db} = \frac{\partial S}{\partial b} + \frac{\partial S}{\partial X_0} \frac{dX_0}{db} + \Lambda_1^T \left( \frac{\partial L_1}{\partial \mathcal{E}'} \frac{\partial \mathcal{E}'}{\partial b} + \frac{\partial L_1}{\partial X_1} \frac{dX_1}{db} \right) \]

\[ + \Lambda_0^T \left( \frac{\partial L_0}{\partial \mathcal{C}'} \frac{\partial \mathcal{C}'}{\partial b} + \frac{\partial L_0}{\partial X_0} \frac{dX_0}{db} + \frac{\partial L_0}{\partial \mathcal{C}'} \frac{dX_1}{db} + \frac{\partial L_0}{\partial X_1} \frac{dX_1}{db} \right). \]
This work uses the numerator layout, by which \( dL_1/db \) is a row vector.

Because the adjoint variables are arbitrary, their values can be selected to cancel out all terms containing state sensitivities, i.e., the derivatives of the states with respect to the design parameters, that is \( dX_1/db \) and \( dX_0/db \). Factorizing eq. (16) and imposing the vanishing of the terms multiplying state sensitivities \( dX_0/db \) and \( dX_1/db \) yields the following governing equations for the adjoint variables,

\[
\begin{align*}
\Lambda^T_1 \frac{\partial L_0}{\partial X_0} + \frac{\partial S}{\partial X_0} &= 0, \quad (17a) \\
\Lambda^T_1 \frac{\partial L_1}{\partial X_1} + \Lambda^T_0 \frac{\partial L_0}{\partial X_1} &= 0. \quad (17b)
\end{align*}
\]

Evaluation of the partial derivatives leads to the adjoint equations,

\[
\begin{align*}
\Xi^T \Lambda_0 + \left( \frac{\partial S}{\partial X_0} \right)^T &= 0, \quad (18a) \\
\Xi^T \Lambda_1 + C^{\prime T} \Lambda_0 &= 0, \quad (18b)
\end{align*}
\]

where \( \Xi^T \), like \( \Xi \) and \( E \), is symmetric. Equation (6) implies that

\[
\frac{\partial S}{\partial X_0} = \text{vec} \left[ \frac{\partial S}{\partial W_0}, \frac{\partial S}{\partial Q_0} \right] = \text{vec} \left[ \frac{\partial S}{\partial W_0}, 0 \right], \quad (19)
\]

where \( 0 \) is a null fourth-order tensor. The system matrices appearing in the adjoint equations are the transposed (and reversed) version of those appearing in the original governing equations, although the right-hand sides differ. Each of the adjoint equations contains \( 6(n+6) \) linear equations, that is, six times the original number of equilibrium equations (4), as well as 36 right-hand side vectors, that is, six times the original number of right-hand sides. This entails an obvious computational burden; however, the cost is independent of the number of design parameters.

The vectorization of the governing equations was necessary to formulate the adjoint problem and solve the linear systems in eq. (18). For the remaining operations, however, the tensor notation is more convenient. Useful expressions and differentiation rules for these fourth-order tensors can be found in mathematics textbooks. The evaluation of eqs. (18a) and (19) calls for the application of the chain rule,

\[
\begin{align*}
\frac{\partial S}{\partial W_0} &= \frac{\partial S}{\partial W_0} + \frac{\partial S}{\partial \alpha} \left( \frac{\partial \alpha}{\partial W_0} + \frac{\partial \alpha}{\partial S} \frac{\partial S}{\partial W_0} \right), \quad (20a) \\
\frac{\partial S}{\partial W_0} &= S_T \begin{bmatrix} -(MZ^T)^T \frac{\partial W}{\partial W_0} \hat{K}^T - (CZ)^T \frac{\partial W}{\partial W_0} \end{bmatrix}, \quad (20b) \\
\frac{\partial S}{\partial \alpha} &= - \hat{K} \frac{\partial \alpha}{\partial \alpha} - \frac{\partial \alpha}{\partial \alpha} \hat{K}^T. \quad (20c)
\end{align*}
\]
Finally, eq. (8a) yields

\[
\frac{\partial \alpha}{\partial W_0} = - \left[ \tilde{K} \frac{\partial W_0^T}{\partial W_0} M + \frac{\partial W_0^T}{\partial W_0} C \right] W_0 - \left[ \left( S_0 Z^T + \tilde{K} W_0^T \right) M + W_0^T C \right] \frac{\partial W_0}{\partial W_0},
\]

(21a)

\[
\frac{\partial \alpha}{\partial S_0} = - \frac{\partial S_0^T}{\partial S_0} Z^T M^T W_0.
\]

(21b)

All terms in eq. (19) have now been evaluated and the final expressions can be vectorized to be inserted into eq. (18a).

4.3 Design sensitivities

Once the adjoint variables have been evaluated using eqs. (18), the design sensitivities (16) become

\[
\frac{dS}{db} = \frac{\partial S}{\partial b} + \Lambda_1^T \frac{\partial E'}{\partial b} + \Lambda_0^T \left( \frac{\partial E'}{\partial b} X_0 + \frac{\partial C'}{\partial b} X_1 + \frac{\partial C'}{\partial b} \right).
\]

(22)

Simplification of this expression gives

\[
\frac{dS}{db} = \frac{\partial S}{\partial b} + \Lambda_1^T \frac{\partial E'}{\partial b} X_1 + \Lambda_0^T \left( \frac{\partial E'}{\partial b} X_0 + \frac{\partial C'}{\partial b} X_1 \right).
\]

(23)

Because each set of \((n+6)\) adjoint equations in eq. (18) is six times singular, the general solution can be stated as \((\Lambda_1 + N \beta)\), where \(\Lambda_1\) is a particular solution, \(N\) is the null space of \(E'\), and \(\beta\) is arbitrary. On the other hand, the product \((\partial E'/\partial b)X_1\) must be orthogonal to \(N\). Consequently, only the projection of \(\Lambda_1\) onto \((\partial E'/\partial b)X_1\) contributes to the sensitivity value, and any particular solution to eq. (18) is valid. Similar reasoning applies to \(\Lambda_0\). In conclusion, although the adjoint variables are not unique, the design sensitivities are.

The evaluation of the sensitivity derivatives in eq. (23) requires the computation of a number of additional derivatives with respect to the design parameters. First, the derivatives of the system matrices must be evaluated based on eqs. (14) to find

\[
\frac{\partial \xi'}{\partial b} = L_6 \otimes \frac{\partial \xi}{\partial b},
\]

(24a)

\[
\frac{\partial \phi'}{\partial b} = L_6 \otimes \frac{\partial \phi}{\partial b},
\]

(24b)

and eqs. (5) now yield

\[
\frac{\partial \xi}{\partial b} = \begin{bmatrix} \partial E/\partial b & \partial C/\partial b Z \\ Z^T(\partial E/\partial b) & \partial C/\partial b \end{bmatrix},
\]

(25a)

\[
\frac{\partial \phi}{\partial b} = \begin{bmatrix} \partial \phi/\partial b - \partial \phi^{T}/\partial b & -\partial M/\partial b \end{bmatrix},
\]

(25b)

The implementation of eq. (23) also requires the evaluation of the partial derivatives of the sectional compliance matrix with respect to the design parameters. For design parameters related
to material properties and fiber orientation angles, the explicit dependency of the compliance matrix on the design parameters is due to the material stiffness matrix $D$. By analogy with eq. (20),

$$\frac{\partial S}{\partial b} = \frac{\partial S_0}{\partial b} + \frac{\partial S}{\partial \alpha} \left( \frac{\partial \alpha}{\partial b} + \frac{\partial \alpha}{\partial S_0} \frac{\partial S_0}{\partial b} \right),$$

(26a)

$$\frac{\partial S_0}{\partial b} = \frac{\partial S_T}{\partial b} \left[ I - (M Z)^T W_0 {\tilde \mathcal{K}}^T - (C Z)^T W_0 \right] + S_T \left[ - \left( \frac{\partial M}{\partial b} Z \right)^T W_0 {\tilde \mathcal{K}}^T - \left( \frac{\partial C}{\partial b} Z \right)^T W_0 \right],$$

(26b)

$$\frac{\partial \alpha}{\partial b} = - \left[ \frac{\partial M}{\partial b} \left( Z S_0 + W_0 {\tilde \mathcal{K}}^T \right) + \frac{\partial C^T}{\partial b} W_0 \right]^T W_0.$$  

(26c)

Sensitivity matrix $\partial S/\partial b$, of size $36 \times n_b$, is obtained as vec($\partial S/\partial b$). Once again, additional derivatives are required. First, the derivatives of the Timoshenko compliance and stiffness matrices

$$\frac{\partial S_T}{\partial b} = \frac{\partial}{\partial b} \left( Z^T M Z \right)^{-1} = - S_T \left( Z^T \frac{\partial M}{\partial b} Z \right) S_T,$$

(27a)

$$\frac{\partial C_T}{\partial b} = \frac{\partial}{\partial b} \left( Z^T M Z \right) = Z^T \frac{\partial M}{\partial b} Z.$$  

(27b)

Finally, the derivatives of the sectional stiffness matrices in eq. (2) expose the explicit dependency on material parameters through sensitivity $\partial D/\partial b$,

$$\frac{\partial M}{\partial b} = \int_A A_L^T \frac{\partial D}{\partial b} A_L \sqrt{g} \, dA,$$

(28a)

$$\frac{\partial C}{\partial b} = \int_A B_L^T \frac{\partial D}{\partial b} B_L \sqrt{g} \, dA,$$

(28b)

$$\frac{\partial E}{\partial b} = \int_A B_L^T \frac{\partial D}{\partial b} R_L \sqrt{g} \, dA.$$  

(28c)

### 4.4 Sensitivities with respect to material properties

The last step of the process is to evaluate the derivatives of the material stiffness matrix with respect to the design parameters. Two simple examples are considered here: (1) the design parameter is the fiber orientation angle within the composite ply, and (2) the design parameter is a material stiffness constant.

If the design parameter is the fiber orientation angle, $\gamma$, eq. (10) yields

$$\frac{\partial D}{\partial \gamma} = \frac{\partial R}{\partial \gamma} D^+ R_T^T + R_T D^+ \frac{\partial R_T}{\partial \gamma}.$$  

(29)
The derivative of rotation matrix $R$, defined by eq. (11) is obtained easily

$$
\frac{\partial R}{\partial \gamma} = \begin{bmatrix}
2\ell'_{1} & m_{1}m'_{1} & 2m_{1}'_{1} & 2m_{1}n_{1}' & 2m_{1}n_{1}' & 2m_{1}n_{1}' & 2m_{1}n_{1}' & 2m_{1}n_{1}' & 2m_{1}n_{1}' & 2m_{1}n_{1}' & 2m_{1}n_{1}' \\
2\ell'_{2} & m_{2}m'_{2} & 2n_{2}m'_{2} & 2m_{2}n_{2}' & 2m_{2}n_{2}' & 2m_{2}n_{2}' & 2m_{2}n_{2}' & 2m_{2}n_{2}' & 2m_{2}n_{2}' & 2m_{2}n_{2}' & 2m_{2}n_{2}' \\
2\ell'_{3} & m_{3}m'_{3} & 2n_{3}m'_{3} & 2m_{3}n_{3}' & 2m_{3}n_{3}' & 2m_{3}n_{3}' & 2m_{3}n_{3}' & 2m_{3}n_{3}' & 2m_{3}n_{3}' & 2m_{3}n_{3}' & 2m_{3}n_{3}' \\
\end{bmatrix}
$$

where notation $(\cdot)'$ indicates a partial derivative with respect to angle $\gamma$ and

$$
R' = [\ell', m', n'] = \begin{bmatrix}
-\sin \gamma & -\cos \gamma & 0 \\
\cos \gamma \cos(\alpha + \beta) & -\sin \gamma \cos(\alpha + \beta) & 0 \\
\cos \gamma \sin(\alpha + \beta) & -\sin \gamma \sin(\alpha + \beta) & 0 \\
\end{bmatrix}
$$

Note that angle $\alpha$ is related to the orientation of the ply within the cross-section, whereas angle $\beta$ vanishes in most practical applications.

If the design parameter is a material stiffness constant, $c$, eq. (10) yields

$$
\frac{\partial D}{\partial c} = R_{x} \frac{\partial D}{\partial \gamma} R_{x}'.
$$

Assuming the design parameter to be the material shear modulus, $c = G_{12}$, the following sensitivity is found,

$$
\frac{\partial D}{\partial G_{12}} = -\frac{\partial D}{\partial c} \frac{\partial c}{\partial G_{12}} = -D^{+} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/G_{12}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/G_{12}^2 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Derivatives with respect to other material constants can be derived in a similar manner.

The proposed strategy can be extended to different types of objective functions and design variables. Geometric parameters, such as those illustrated in fig. [5] are particularly interesting but require a special treatment: indeed, changes in the geometry of the section affect the finite element mesh, which then enters the computation of the sensitivities. This important aspect of the problem is an active research topic.

### 5 Numerical results

Typically, the proposed approach to the evaluation of sensitivity derivatives is embedded within a gradient-based structural optimization tool. The numerical results presented in this section focus on the accuracy and efficiency of the sensitivity analysis. To
verify the implementation, three approaches will be used to compute the design sensitivities of sectional compliance matrices: the adjoint variable method presented above, the complex-step differentiation \cite{52,53}, and the real-step or finite-difference differentiation. The examples presented below use fiber orientation angles as design variables. Complex-step perturbations are selected to be \(\Delta \gamma = \pi/2 \times 10^{-15}\) rad. The real-step method uses the central finite difference approach with perturbations \(\Delta \gamma = \pi/2 \times 10^{-7}\) rad.

5.1 C-section

The first problem to be investigated consists of a straight cantilevered beam with the C-shaped cross-section shown in fig. 6. The configuration of the cross-section is as follows: web height, \(h_w = 1.6\) m, flange width, \(w_f = 0.8\) m, web thickness, \(t_w = 0.1\) m, and flange thickness, \(t_f = 0.05\) m. The beam is made of a transversely isotropic material with the following properties: \(E_1 = 180\) GPa, \(E_2 = 10\) GPa, \(\nu_{12} = \nu_{23} = 0.28\), \(G_{12} = 7\) GPa, and \(\gamma = -45\) degrees. The total number of degrees of freedom is \(n = 198\).

Using SectionBuilder, the warping field, \(W\), depicted in fig. 7 was found, and the corresponding sectional stiffness matrix is

\[
\frac{K}{10^9} = \begin{bmatrix}
3.759 & -0.031 & 0.000 & -0.000 & -0.000 & -0.644 \\
-0.031 & 0.294 & -0.000 & 0.000 & 0.000 & 0.002 \\
0.000 & -0.000 & 0.645 & -0.108 & -0.018 & -0.000 \\
-0.000 & 0.000 & -0.108 & 0.021 & 0.003 & -0.000 \\
-0.000 & 0.000 & -0.018 & 0.003 & 1.247 & -0.000 \\
-0.644 & 0.002 & -0.000 & 0.000 & 0.000 & 0.288
\end{bmatrix},
\]  

(34)

where the units of the axial and shearing components are [N], the units of the torsion and bending components are [N·m²], and the units of the remaining entries are [N·m]. Due to the configuration of the cross-section and the presence of anisotropic composite materials, the stiffness matrix is fully populated.

The sensitivity derivative of the sectional stiffness matrix with respect to the fiber orientation angle, \(dK/d\gamma\), was evaluated. This sensitivity can be expressed in terms of those of the sectional compliance matrix,

\[
\frac{dK}{d\gamma} = \frac{dS^{-1}}{d\gamma} = -\frac{K}{S} \frac{dS}{d\gamma} K,
\]  

(35)

which gives

\[
\frac{dK}{d\gamma} \frac{1}{10^9} = \begin{bmatrix}
5.826 & -0.097 & -0.000 & -0.000 & -0.000 & -0.996 \\
-0.097 & 0.188 & -0.000 & 0.000 & 0.000 & 0.006 \\
-0.000 & -0.000 & 0.369 & -0.061 & -0.051 & 0.000 \\
-0.000 & 0.000 & -0.061 & 0.012 & 0.009 & -0.000 \\
0.000 & -0.000 & -0.051 & 0.009 & 1.936 & -0.000 \\
-0.996 & 0.006 & -0.000 & 0.000 & 0.000 & 0.451
\end{bmatrix},
\]  

(36)

Note that its units are the units of \(K\) divided by [rad]. Generally, the largest components of the stiffness matrix exhibit the largest sensitivity with respect to the fiber orientation angle (see, for instance, terms \(dK_{11}/d\gamma\), \(dK_{55}/d\gamma\) and \(dK_{16}/d\gamma\)). However, the sensitivity of some off-diagonal
Figure 7: C-section warping modes under axial, 2-shear, 3-shear, torsion, 2-bending and 3-bending loads

terms is amplified (see terms $\frac{dK_{12}}{d\gamma}$) or reduced (see $\frac{dK_{34}}{d\gamma}$). This highlights the importance evaluating the sensitivity of all entries of the compliance matrix, including those that capture the coupling between the different deformation modes. Increasing the fiber orientation angle $\gamma$, i.e., making it less negative, increases the diagonal terms of the stiffness matrix, thus resulting in positive sensitivity values, which is consistent with the physics of the problem.

Table 1 shows the six diagonal entries of matrix $\frac{dK}{d\gamma}$. For the purposes of code verification, Table 1 also lists the predictions of the complex-step differentiation and real-step finite difference procedures. Note the very good agreement between the complex-step differentiation, which provides exact sensitivities in the limit of an infinitesimal perturbation size, and the adjoint method. Because the complex-step differentiation does not suffer from roundoff errors [53], the discrepancy between the adjoint method and the complex-step differentiation is very small, of the order of the machine accuracy. On the other hand, the real-step finite difference is very sensitive to roundoff errors and hence, a larger perturbation size must be used, leading to much larger errors in the sensitivity derivatives.

To further assess the accuracy of the proposed adjoint approach, a convergence study was performed. For various mesh sizes (i.e., numbers of degrees of freedom, $n$), the sectional stiffness matrix and its sensitivity derivatives were evaluated and the results were compared with those obtained using a very fine reference mesh. The convergence was measured through the following
Table 1: Diagonal sensitivities, along with relative error (C-section)

<table>
<thead>
<tr>
<th>Component</th>
<th>Method</th>
<th>Value</th>
<th>Rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>d(K_{11}/d\gamma) [N/rad]</td>
<td>Complex</td>
<td>5.82660533572841 (\times 10^{+09})</td>
<td>3 (\times 10^{-15})</td>
</tr>
<tr>
<td></td>
<td>Adjoint</td>
<td>5.82660533572840 (\times 10^{+09})</td>
<td>2 (\times 10^{-08})</td>
</tr>
<tr>
<td></td>
<td>Real</td>
<td>5.82660546385765 (\times 10^{+09})</td>
<td></td>
</tr>
<tr>
<td>d(K_{22}/d\gamma) [N/rad]</td>
<td>Complex</td>
<td>1.88431565811246 (\times 10^{+08})</td>
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<td>Adjoint</td>
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</tr>
<tr>
<td></td>
<td>Real</td>
<td>1.88431824801110 (\times 10^{+08})</td>
<td>1 (\times 10^{-06})</td>
</tr>
<tr>
<td>d(K_{33}/d\gamma) [N/rad]</td>
<td>Complex</td>
<td>3.69099760965652 (\times 10^{+08})</td>
<td>5 (\times 10^{-13})</td>
</tr>
<tr>
<td></td>
<td>Adjoint</td>
<td>3.69099760965849 (\times 10^{+08})</td>
<td>7 (\times 10^{-08})</td>
</tr>
<tr>
<td></td>
<td>Real</td>
<td>3.69099733703099 (\times 10^{+08})</td>
<td></td>
</tr>
<tr>
<td>d(K_{44}/d\gamma) [N\cdot m²/rad]</td>
<td>Complex</td>
<td>1.2050745595739 (\times 10^{+07})</td>
<td>6 (\times 10^{-12})</td>
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</tr>
<tr>
<td></td>
<td>Real</td>
<td>1.20508972294757 (\times 10^{+07})</td>
<td></td>
</tr>
<tr>
<td>d(K_{55}/d\gamma) [N\cdot m²/rad]</td>
<td>Complex</td>
<td>1.93634214035252 (\times 10^{+09})</td>
<td>4 (\times 10^{-15})</td>
</tr>
<tr>
<td></td>
<td>Adjoint</td>
<td>1.93634214035251 (\times 10^{+09})</td>
<td>3 (\times 10^{-10})</td>
</tr>
<tr>
<td></td>
<td>Real</td>
<td>1.93634213982217 (\times 10^{+09})</td>
<td></td>
</tr>
<tr>
<td>d(K_{66}/d\gamma) [N\cdot m²/rad]</td>
<td>Complex</td>
<td>4.51814919675647 (\times 10^{+08})</td>
<td>3 (\times 10^{-14})</td>
</tr>
<tr>
<td></td>
<td>Adjoint</td>
<td>4.51814919675632 (\times 10^{+08})</td>
<td>2 (\times 10^{-07})</td>
</tr>
<tr>
<td></td>
<td>Real</td>
<td>4.51814996156295 (\times 10^{+08})</td>
<td></td>
</tr>
</tbody>
</table>

indices:

\[ \epsilon_K = \frac{\|K - \overline{K}\|}{\|K\|}, \quad \epsilon_S = \frac{\left\|\frac{dK}{d\gamma} - \frac{dK}{d\gamma}^T\right\|}{\left\|\frac{dK}{d\gamma}\right\|}, \tag{37} \]

where \(\overline{K}\) and \(d\overline{K}/d\gamma\) are the reference sectional stiffness matrix and its sensitivity derivatives, respectively. Figure 8 shows the error index as a function of \(\hat{n}\); the dashed lines correspond to the present C-section. The sectional stiffness matrices of open sections are known to be strongly affected by the warping field \(\hat{c}\). Consequently, the convergence rate is poor for this type of problems.

The sectional compliance matrix is obtained from eq. (8c); note that the terms on the right-hand side of the equations do not present a symmetric structure. The sectional compliance matrix, however, must be symmetric if all equations are solved exactly \(\hat{c}\). Hence, the lack of symmetry of the stiffness and sensitivity matrices can be quantified through indices

\[ \rho_K = \frac{\|K - \overline{K}^T\|}{\|K\|}, \quad \rho_S = \frac{\left\|\frac{dK}{d\gamma} - \left(\frac{dK}{d\gamma}\right)^T\right\|}{\left\|\frac{dK}{d\gamma}\right\|}, \tag{38} \]
which are a good measure of the accuracy of the numerical process. These indices are shown in fig. 9 as a function of the mesh size, \( n \). As expected, similar levels of numerical accuracy are obtained for the evaluation of the stiffness matrix and of its sensitivity derivatives, further validating the implementation of the adjoint method.

![Figure 8: Relative error in C-section (CS) and Double-box (DB)](image8)

![Figure 9: Symmetry error in C-section (CS) and Double-box (DB)](image9)

5.2 Double-box section

The second example consists of a straight beam with the double-box cross-section depicted in fig. 10. The dimensions are as follows: \( w_f = 1 \text{ m}, t_f = 0.08 \text{ m}, h_w = 1 - 2t_f, t_w = 0.3 \text{ m}, t_o = 0.04 \text{ m} \). The beam is made of a transversely anisotropic material with the following stiffness properties: \( E_1 = 180 \text{ GPa}, E_2 = 10 \text{ GPa}, \nu_{12} = \nu_{23} = 0.28, G_{12} = 7 \text{ GPa} \). The lay-up stacking sequence for the various components of the section are as follows: lower flange with four-ply lay-up at \( \gamma_{1-4} = [0, -45, +45, 0] \) degrees, upper flange with four-ply lay-up at \( \gamma_{5-8} = [0, +45, -45, 0] \) degrees, left web with single-ply lay-up at \( \gamma_9 = [0] \) degrees, main web with eight-ply lay-up at \( \gamma_{10-17} = [0, -45, +90, 0, 0, +90, -45, 0] \) degrees, and right web with single-ply lay-up at \( \gamma_{18} = 0 \) degrees. The total number of degrees of freedom is \( n = 981 \).

SectionBuilder was run to evaluate the warping field, \( W \), depicted in fig. 11 and the corresponding sectional stiffness matrix is

\[
\begin{bmatrix}
5.225 & -0.005 & 0.000 & 0.002 & 2.612 & -2.612 \\
-0.005 & 0.097 & -0.000 & -0.048 & -0.002 & 0.000 \\
0.000 & -0.000 & 0.196 & 0.098 & 0.000 & -0.000 \\
0.002 & -0.048 & 0.098 & 0.108 & 0.002 & -0.001 \\
2.612 & -0.002 & 0.000 & 0.002 & 1.855 & -1.306 \\
-2.612 & 0.002 & -0.000 & -0.001 & -1.306 & 1.737
\end{bmatrix}
\]

where, again, the units of the axial and shearing components are \([\text{N}]\), the units of the torsion and bending components are \([\text{N}\cdot\text{m}^2]\), and the units of the remaining entries are \([\text{N}\cdot\text{m}]\). The sectional stiffness matrix is also fully populated due to the presence of anisotropic composite materials, underlining the need to use a finite element based tool for the sectional analysis.
Table 2 shows the six largest entries of matrix $\frac{dK}{d\gamma}$, evaluated using the proposed adjoint method. The table also lists the predictions of the complex- and real-step differentiation approaches and their relative error. Here again, the approximate solutions provided by the complex- and real-step differentiation approaches are in good agreement with the adjoint solution, although the accuracy of the real-step differentiation approach is mediocre at best.

The sensitivities with respect to angles $\gamma_{16}$, $\gamma_{11}$, $\gamma_{14}$, and $\gamma_{13}$ are the largest, which is consistent with the symmetry of the cross-section (see fig. 10). The two largest values ($\frac{dK_{11}}{d\gamma_{16}}$ and $\frac{dK_{11}}{d\gamma_{11}}$) are associated with the axial stiffness $K_{11}$, which is the largest source of stiffness in this cross-section. Furthermore, increasing the value of $\gamma_{16}$ and $\gamma_{11}$, i.e., making them less negative, also increases their axial stiffness, hence the positive sign of the sensitivities.

Figure 8 shows the convergence study for the sectional stiffness matrix and its sensitivity derivatives as functions of the mesh size, $n$. The results for the double-box are displayed as solid lines. Because the double-box is a closed section, the convergence rate is faster than that observed for the (open) C-section problem. The overall accuracy of the computation as measured by the symmetry index, shown in fig. 9, is also better than that observed for the C-section.

The computations presented here were performed on an Intel(R) Xeon(R) CPU at 3.5 GHz with 16 GB RAM. The computation times are shown in table 3 along with the runtime relative to the adjoint variable method runtime. The adjoint method is computationally more efficient than its numerical counterparts. Note that these results are preliminary, as they were obtained on a serial MATLAB implementation. Similar efficiency trends are expected in the upcoming C/C++ implementation, though a parallel implementation might alter the time distribution.

For cases involving larger numbers of design variables, the accuracy is expected to remain the same. The performance, on the other hand, deserves a closer scrutiny. The adjoint variable method requires only one solution of the equations (4), no matter how many design parameters are involved, as opposed to direct methods that require one system solution per design parameter. The cost of
Figure 11: Double-box warping modes under axial, 2-shear, 3-shear, torsion, 2-bending and 3-bending loads

evaluating the partial derivatives in eqs. (26) and (28) is, however, proportional to the number of design parameters. Nevertheless, this computational overhead is far smaller than that associated with the assembly and solution of the equations (4).

6 Conclusions

This paper has presented an approach to the computation of the design sensitivities of the sectional properties of composite beams based on the adjoint method. This is a fundamental step toward the design and optimization of flexible multibody systems involving beam elements with anisotropic cross-sections. Such level of detail is rarely seen in the flexible multibody systems literature.

The paper has summarized the governing equations for the sectional analysis: the warping field is evaluated first based on a finite element analysis of the cross-section, and the sectional compliance matrix is a byproduct of this sectional analysis. Next, the adjoint and design sensitivity equations were presented. Because the cross-sectional analysis formulation presents unique characteristics, the resulting adjoint method also presents a unique structure. All intermediate derivatives and higher-order tensors stemming from the use of the sectional analysis were listed in the paper.

Finally, the approach was verified by comparing its predictions with those of the complex- and real-step numerical differentiation methods. The sensitivity derivatives obtained from the proposed adjoint approach were found to be in good agreement (to machine precision) with those found with the complex-step method; the accuracy of the sensitivities obtained from the real-step method was
Table 2: Largest sensitivities, along with relative error (double-box)

<table>
<thead>
<tr>
<th>Component</th>
<th>Method</th>
<th>Value</th>
<th>Rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dK_{11}/d\gamma_{16}$ [N/rad]</td>
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<td></td>
<td>Adjoint</td>
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<td>$dK_{11}/d\gamma_{7}$ [N/rad]</td>
<td>Complex</td>
<td>$4.95789928782145 \times 10^{+08}$</td>
<td>$3 \times 10^{-14}$</td>
</tr>
<tr>
<td></td>
<td>Adjoint</td>
<td>$4.95789928782133 \times 10^{+08}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Real</td>
<td>$4.95789711785936 \times 10^{+08}$</td>
<td>$4 \times 10^{-07}$</td>
</tr>
</tbody>
</table>

questionable. Furthermore, the adjoint method showed good convergence with respect to the mesh size, and the symmetry of the stiffness matrix derivative was preserved.

Future research will focus (1) on the evaluation of sensitivity derivatives with respect to geometric parameters, and (2) on the coupling of sectional and beam analysis tools to address the design and optimization of flexible multibody systems involving beam elements.

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Table 3: Performance of the different sensitivity analysis methods (double-box)

<table>
<thead>
<tr>
<th>Method</th>
<th>Runtime (s)</th>
<th>Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adjoint</td>
<td>19.78</td>
<td>1.00×</td>
</tr>
<tr>
<td>Complex</td>
<td>144.95</td>
<td>7.33×</td>
</tr>
<tr>
<td>Real</td>
<td>281.67</td>
<td>14.24×</td>
</tr>
</tbody>
</table>
as Dr. Shilei Han for the useful input.
References


