On the Analysis of Periodically Heterogenous Beams*

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Abstract

Based on the symplectic transfer-matrix method, this paper develops a novel approach to the analysis of beams presenting periodic heterogeneities along their span. The approach, rooted in the Hamiltonian formalism, generalizes developments presented earlier by the authors for span-wise uniform beams. Starting from the kinematics of a unit cell, the approach proceeds through a set of structure-preserving symplectic transformations and decomposes the solution into its central and extremity components. The geometric configuration and material properties of the unit cell may be arbitrarily complex as long as the cell’s two end cross-sections are identical. The proposed approach identifies an equivalent, homogenized beam with uniform curvatures and sectional stiffness characteristics along its span. Numerical examples are presented to demonstrate the capabilities of the analysis. Predictions are found to be in excellent agreement with those obtained by full finite element analysis.

1 Introduction

This paper focuses on beam-like structures exhibiting periodically varying geometric configuration and material properties along their span. The beam is composed of identical cells, referred to as “periodic cells” or “unit cells,” that repeat periodically along the beam’s span and are interconnected through identical cross-sections. A finite element model of the complete structure might lead to considerable computational effort if the periodic nature of the structure is not taken into account. The goal of this paper is to develop a beam theory that takes advantage of the characteristics of the problem and leads to a more efficient analysis.

Let \( L \) and \( a \) be the lengths of the beam and of a representative dimension of the cross-section, respectively. If the length, \( L_c \), of a periodic cell is smaller than that of the beam, the problem presents two small parameters, \( a/L \) and \( L_c/L \), which have been used to reduce the structure to a one-dimensional beam through two-scale asymptotic analysis. For instance, Kolpakov [1] and Kalamkarov et al. [2] developed systematic asymptotic methods for the analysis of periodically heterogenous beams. By using a two-scale asymptotic expansion, the three-dimensional elasticity problem was split into a local problem posed on a unit cell, which provides the beam’s \( 4 \times 4 \) sectional stiffness matrix, and a global, one-dimensional homogenized beam problem. Yi et al. [3] generalized Kolpakov’s method by using a finite element implementation. Similar approaches were also developed by Buannic and Cartraud [4, 5], where appropriate boundary conditions were derived by matching the outer solution with the analytical decaying end solutions. Their approach was extended to tackle helicoidal beam problems by Messager and Cartraud [6]. Asymptotic approaches

yield accurate results when $L_c/L \ll 1$ but their predictions deteriorate as the length of the unit cell increases.

Exact solutions of complete periodic beam problems can be determined through the analysis of a unit cell only. A fundamental challenge of this approach is to identify the algebraic structure of the corresponding solutions. For periodic trusses, Renton [7] found that the characteristic solutions consist of simple polynomial functions and exponentially decaying solutions for the non-vanishing and vanishing stress resultants, respectively. Moreover, the polynomial solutions yield the beam’s equivalent stiffness properties, akin to its axial, torsional, bending, and shear stiffnesses. Stephen and Wang [8, 9] developed a symplectic transfer-matrix method for the analysis of periodic trusses. The stiffness characteristics of a unit cell are characterized by a symplectic transfer matrix, whose eigenvalues and eigenvectors yield the beam’s equivalent stiffness properties.

A closely related problem is the analysis of span-wise uniform beams. Mielke [13, 14] found that the solutions of Saint-Venant’s problem lie in a center manifold spanned by the twelve generalized eigenvectors associated with the null eigenvalues. These eigenvectors form four Jordan chains; six of them correspond to the beam’s rigid-body modes while the others are the fundamental deformation modes of the beam: extension, torsion, and bending and shearing in two directions. Zhong [15, 16] developed novel analytical techniques for two-dimensional uniform strips and three-dimensional uniform cylinders based on Hamilton’s formalism. The stiffness of the structure is characterized by a Hamiltonian matrix whose null and purely imaginary eigenvalues give rise to the solution of Saint-Venant’s problem. Its eigenvalues with non-vanishing real parts give rise to decaying solutions and the associated characteristic decay lengths provide a quantification of Saint-Venant’s principle. The Jordan chain structure of Saint-Venant’s problem for planar strips and cylinders was also identified by Zhong [15, 16] and Druz and Ustinov [17]. Bauchau and Han [18] developed an approach to the solution of Saint-Venant’s problem based on Hamilton’s formalism. This approach proceeds through a sequence of structure preserving transformations using symplectic matrices and decomposes the solution into its central (Saint-Venant’s) and extremity (decaying end) components. In this approach, the explicit construction of the Jordan form is avoided. The same authors further generalized the approach to beams undergoing large motion but small strains [19] and helicoidal beam problems [20].

A span-wise uniform beam can always be regarded as a periodic beam for which the cell length is arbitrary. As shown by Zhong [21] and Stephen [9], the symplectic transfer matrix is identical to the state transition matrix of the corresponding Hamiltonian system. Hence, the transfer-matrix approach is able to predict the beam’s sectional stiffness and to recover the local three-dimensional stress field through a static analysis of a slice of the beam. The symplectic transfer-matrix approach is based on the static analysis of a unit cell that can be performed by commercial finite element packages. Consequently, the approach presented in this paper can be implemented without developing new finite element formulations.

This paper presents a novel strategy for the analysis of periodically heterogenous beams based on the symplectic transfer-matrix approach, which parallels the Hamiltonian approach developed by the authors [18, 19, 20]. The proposed method is valid for periodically heterogenous beams with arbitrarily complex unit cells, as long as the two end cross-sections of the periodic cell are identical. The unit cells need not be slender, beamlike structures; they could be truly three-dimensional structures. Central and extremity solutions are found based on structure preserving transformations of the symplectic transfer matrix. The proposed method leads to an equivalent sectional stiffness matrix of size $6 \times 6$ for periodic beams. The associated boundary conditions highlight the interaction
between the central and extremity solutions.

The paper is organized as follows: the kinematics and governing equations of the problem are presented in sections 2 and 3, respectively. The construction of central and extremity solutions for a unit cell are discussed in sections 4 and 5, respectively. Section 6 discusses the complete solution for periodically heterogenous beams and the corresponding boundary conditions. Illustrative numerical examples are presented in section 7.

## 2 Kinematics of the problem

Figure 1 depicts a periodically heterogenous beam composed of \( m \) identical cells that repeat periodically along the beam’s reference line and are interconnected through identical cross-sections. The unit cells are of arbitrary shape and present arbitrary material heterogeneities but are all identical. As illustrated in the figure, they could be truly three-dimensional structures. The cross-sections that connect the periodic cells are denoted \( s_k \), \( k = 1, 2, \ldots m+1 \). All are identical and cross-sections \( k \) and \( k+1 \) are the end sections of unit cell \( k \).

![Figure 1: A typical periodically heterogenous beam.](image)

### 2.1 Kinematics of a unit cell

As depicted in fig. 2, cross-sections \( s_k \) and \( s_{k+1} \) lie in planes defined by mutually orthogonal unit vectors \((\vec{b}_k^1, \vec{b}_k^2, \vec{b}_k^3)\) and \((\vec{b}_{k+1}^1, \vec{b}_{k+1}^2, \vec{b}_{k+1}^3)\), respectively, and by two reference points denoted \( B_k \) and \( B_{k+1} \), respectively. Two reference frames are attached to these cross-sections, \( F_k = [B_k, F_k] = ([\vec{b}_k^1, \vec{b}_k^2, \vec{b}_k^3]) \) and \( F_{k+1} = [B_{k+1}, F_{k+1}] = ([\vec{b}_{k+1}^1, \vec{b}_{k+1}^2, \vec{b}_{k+1}^3]) \) for sections \( s_k \) and \( s_{k+1} \), respectively.

The motion tensors [22] that bring inertial frame \( F_I = [O, I = (\vec{i}_1, \vec{i}_2, \vec{i}_3)] \) to sectional frames \( F_k \) and \( F_{k+1} \) are

\[
\mathcal{C}_k = \begin{bmatrix} R_k & \vec{r}_k R_k \\ 0 & R_k \end{bmatrix}, \quad \mathcal{C}_{k+1} = \begin{bmatrix} R_{k+1} & \vec{r}_{k+1} R_{k+1} \\ 0 & R_{k+1} \end{bmatrix},
\]

respectively, where \( \vec{r}_k \) and \( \vec{r}_{k+1} \) are the position vectors of points \( B_k \) and \( B_{k+1} \) with respect to point \( O \), respectively, and \( R_k \) and \( R_{k+1} \) are the rotation tensors that bring inertial basis \( I \) to sectional bases \( B_k \) and \( B_{k+1} \), respectively. The relative motion tensor that brings frame \( F_k \) to \( F_{k+1} \) is defined as

\[
\mathcal{R} = \mathcal{C}_k^{-1} \mathcal{C}_{k+1} = \begin{bmatrix} R_k & \vec{r}_I R_k \\ 0 & R_{k+1} \end{bmatrix},
\]

![Figure 2: Configuration of the unit cell](image)
where $R^{[k]} = R_{k}^T R_{k+1}^T$, $C^{[k]} = R_{k}^T (L_{k+1} - L_k)$, and notation $(\cdot)^{[k]}$ indicates tensor components resolved in frame $F_{k}$. The components of relative motion tensor $\bar{R}$ are identical in frames $F_k$ and $F_{k+1}$, i.e., $\bar{R} = \bar{R}^{[k]} = \bar{R}^{[k+1]}$.

For convenience, the relative motion tensor will be brought to its canonical form through a change of frame operation, $\bar{R} = \bar{C}^{-1} \bar{R} \bar{C}$, where motion tensor $\bar{C}$ is

$$\bar{C} = \begin{bmatrix} \bar{R} & \bar{s} \bar{R} \bar{R} \\ 0 & 0 \end{bmatrix}. \tag{3}$$

Rotation tensor $\bar{R}$ and vector $\bar{s}$ are selected to bring the relative motion tensor to its canonical form,

$$\bar{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 & -d \sin \phi & -d \cos \phi \\ 0 & \sin \phi & \cos \phi & 0 & d \cos \phi & -d \sin \phi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & 0 & 0 & \sin \phi & \cos \phi \end{bmatrix}, \tag{4}$$

where $d = \bar{n}^T \bar{L}^{[k]}$ is the intrinsic displacement of frame $F_{k+1}$ with respect to $F_k$, $\phi$ the rotation of basis $B_{k+1}$ with respect to $B_k$, and $\bar{n}$ the unit vector about which this rotation takes place. Simple algebra reveals that $(2 \sin \phi/2) \bar{s} = [\cos(\phi/2) \bar{n} - \sin(\phi/2) \bar{m} \bar{n}] \bar{L}^{[k]}$ and rotation tensor $\bar{R} = [\bar{n} \bar{i} \bar{b}]$, where mutually orthogonal unit vectors $\bar{i}$ and $\bar{b}$ can be selected arbitrarily in the plane perpendicular to $\bar{n}$.

### 2.2 Change of frame relationships

Motion tensors express change of frame relationships [22]. Consider an infinitesimal rigid-body motion of the unit cell denoted $U^T = \{u_O, \phi^T\}$, where $u_O$ are the components of the displacement vector of the material point of the cross-section whose location coincides with that of inertial point $O$ and $\phi$ the components of rigid-body rotation of the cell, both resolved in inertial basis $I$. The components of the displacement vector of material point $B^k$ is $u_B = u_O + \bar{r} \phi$ and its components in basis $B_k$ are $u_B^{[k]} = \bar{R}^T u_O - \bar{R}^T \bar{r} \phi$. If follows that $U^{[k]} = \bar{C}^{-1} U$, where $U^{[k]} = \{u_B^{[k]} T, \phi^{[k]} T\}$ gives the components of the rigid-body motion resolved in frame $F_k$. A similar reasoning yields $U^{[k+1]} = \bar{C}^{-1} \bar{C}_k U$, where array $U^{[k+1]}$ stores the components of the rigid-body motion resolved in frame $F_{k+1}$ and finally,

$$U^{[k]} = \bar{R} U^{[k+1]} \tag{5}.$$ 

Clearly, motion tensors perform change of frame operations.

Consider now a loading applied to the cross-section, $F^T = \{F_O, M_O^T\}$, where $F_O$ is the force vector and $M_O$ the moment resultant with respect to origin $O$, both resolved in inertial basis $I$. The equipollent loading acting at material point $B^k$ is $F_B = F_O$ and $M_B = M_O - \bar{r} \bar{r} F_B$. When resolved in frame $F_k$, this equipollent loading becomes $F^{[k]} = \bar{C}_k F$, where $F^{[k]} T = \{F_B^{[k]} T, M_B^{[k]} T\}$. A similar reasoning yields $F^{[k+1]} = \bar{C}^T_{k+1} F$. Finally,

$$F^{[k]} = \bar{R}^{-T} F^{[k+1]} \tag{6}.$$ 

Equations (5) and (6) provide the change of frame formulæ for displacement and forces, respectively; as expected, work is a frame indifferent quantity, i.e., $U^T F = U^{[k]} T F^{[k]} = U^{[k+1]} T F^{[k+1]}$. 


2.3 Rigid-body motion

Consider a finite element mesh of the unit cell \( k \) for which the meshes of the two end sections, \( s_k \) and \( s_{k+1} \), are identical and let arrays \( \hat{u}_k \) and \( \hat{u}_{k+1} \) denote the nodal displacement arrays for these two sections, respectively. Let \( N \) be the number of nodes used to discretize the two sections. The common size of these arrays, denoted \( n \), depends on the node type: \( n = 3N \) for brick or truss nodes and \( n = 6N \) for beam or shell nodes.

Let array \( U^T = \{ u^T, \phi^T \} \) denote a infinitesimal rigid-body motion of the cell. Its components resolved in frames \( F_k \) and \( F_{k+1} \) are

\[
U[k] = C^{-1}u \quad \text{and} \quad U[k+1] = C^{-1}_{k+1}u,
\]

as discussed in section 2.2. Let \( q \) denote the relative position vector of a material point located on cross-section \( s_k \) with respect to point \( B_k \), as depicted in fig. 2. Under rigid-body motion \( U \), the displacement field over cross-section \( s_k \) is

\[
u[k] = \tilde{z} U[k],\]

where the expression for matrix \( \tilde{z} \) depend on the node type

\[
\tilde{z} = \begin{cases} 
I - \tilde{q}[k], & \text{for brick or truss nodes;} \\
I - \tilde{q}[k], & \text{for beam or shell nodes.}
\end{cases}
\]

Matrix \( I \) is the identity matrix of size \( 3 \times 3 \) and array \( q[k] \) stores the components of vector \( q \) resolved in basis \( B_k \). Matrix \( Z \), of size \( n \times 6 \), stacks the rows of matrix \( \tilde{z} \) for each of the nodes at the cross-section, \( s_k \). Because the physical configuration and meshes of cross-sections \( s_k \) and \( s_{k+1} \) are identical, \( Z \) is identical for these two sections, and hence, is identical for all sections.

3 Governing equations

Span-wise uniform beams can be viewed as heterogeneous beams with unit cells of arbitrary length. Hence, the solution of homogeneous problem is reviewed in section 3.1 before addressing periodic problems in section 3.2.

3.1 Uniform beam problems

Bauchau and Han [18, 19] showed that the governing equations for span-wise uniform beams can be cast in the following form,

\[
\dot{X}^T = H \dot{X}, \tag{8}
\]

where array \( X^T = \{ \dot{u}^T, \dot{P}^T \} \) stores the nodal displacements and forces, notation \( (\cdot)' \) indicates partial derivatives with respect to the arc-length coordinate, \( \alpha_1 \), along the beam’s reference line and \( H \) is a Hamiltonian matrix characterizing the sectional stiffness of the beam. The null and purely imaginary eigenvalues of the Hamiltonian give rise to polynomial and trigonometric solutions, respectively, corresponding to the solution of Saint-Venant’s problem. Its eigenvalues presenting non-vanishing real parts give rise to exponentially decaying solutions.

Solutions for uniform beam problems are found by integrating eq. (8) to find \( \dot{X} (\alpha_1) = \exp(\alpha_1H) \dot{X}_0 \), where \( \dot{X}_0 \) are the boundary conditions at \( \alpha_1 = 0 \). It follows that \( \dot{X}_1 = S \dot{X}_0 \), where \( \dot{X}_1 = \dot{X} (\alpha_1 = L) \) is the solution of the problem at \( \alpha_1 = L \) and \( S = \exp(LH) \) is a symplectic matrix because the exponential of a Hamiltonian matrix is symplectic. Matrix \( S \), referred to as the “transfer matrix,” expresses the relationship between the solutions of the problem at \( \alpha_1 = 0 \) and \( L \). The transfer-matrix method is widely used in the analysis of wave propagation, rotor dynamics, structural analysis, and the development of time integration schemes.
The close relationship between the transfer and Hamiltonian matrices is a key feature of this paper,

\[ H = \frac{1}{L} \log(S) \iff S = \exp(LH). \]  

(9)

Mathematically, the members of the Lie algebra of the symplectic group are Hamiltonian matrices.

In classical mechanics, solutions of Hamilton’s canonical equations can be represented by symplectic transfer matrices [23]. In elasticity, the connection between the transfer-matrix method and the Hamiltonian formulation was identified by Zhong [15, 16, 21] and Karpov and Stephen [24, 25, 9]. The symplectic transfer-matrix approach does not require span-wise uniform beams, rather, it remains valid for arbitrary unit-cell configurations as long as its end cross-sections remain identical. The transfer-matrix method leads to the definition of an equivalent beam model that will be discussed in section 4.5.

Given the close relationship between the Hamiltonian and transfer matrices expressed by eq. (9), the following conclusions can be reached: the unit eigenvalues of the symplectic transfer matrix give rise to the solution of Saint-Venant’s problem, while the reciprocal pairs of eigenvalues give rise to exponentially decaying solutions. These facts form the basis for the analysis of straight, twisted, and planar curved periodic beam-like trusses by Stephen and his coauthors [24, 11, 9]. In this paper, the symplectic transfer-matrix method is used for the analysis more complex heterogenous beam structures.

3.2 Heterogeneous beam problems

Figure 3 sketches a three-dimensional mesh of the unit cell of a periodic beam. Any commercial FEM code can evaluate the stiffness matrix of this structure and using static condensation, all internal degrees of freedom can be eliminated. The strain energy, \( V \), of the cell can then be expressed in terms of the degrees of freedom of the end sections, \( s_k \) and \( s_{k+1} \), only,

\[ V = \frac{1}{2} \left\{ \begin{array}{c} \hat{u}_k \\ \hat{u}_{k+1} \end{array} \right\}^T \begin{bmatrix} K_{k,k} & K_{k,k+1} \\ K_{k+1,k} & K_{k+1,k+1} \end{bmatrix} \left\{ \begin{array}{c} \hat{u}_k \\ \hat{u}_{k+1} \end{array} \right\}, \]  

(10)

Figure 3: The internal and retained DOFs of a unit cell.

where \( K_{k,k}, K_{k,k+1} \) and \( K_{k+1,k+1} \) are the reduced stiffness matrices. Typically, all nodal displacement components are resolved in inertial basis \( I \). For the present development, it is necessary to resolve sectional displacements \( \hat{u}_k \) and \( \hat{u}_{k+1} \) in sectional bases \( \mathcal{B}_k \) and \( \mathcal{B}_{k+1} \), respectively, i.e., \( \hat{u}_k = \mathcal{R}_k \hat{u}_k^{[k]} \) and \( \hat{u}_{k+1} = \mathcal{R}_{k+1} \hat{u}_{k+1}^{[k]} \), where \( \mathcal{R}_k = \text{diag}(R_k, \ldots, R_k) \) and \( \mathcal{R}_{k+1} = \text{diag}(R_{k+1}, \ldots, R_{k+1}) \). Strain energy (10) now becomes

\[ V = \frac{1}{2} \left\{ \begin{array}{c} \hat{u}_k^{[k]} \\ \hat{u}_{k+1}^{[k]} \end{array} \right\}^T \begin{bmatrix} \mathcal{R}_k^T K_{k,k} \mathcal{R}_k \\ \mathcal{R}_{k+1}^T K_{k+1,k+1} \mathcal{R}_{k+1} \end{bmatrix} \left\{ \begin{array}{c} \hat{u}_k^{[k]} \\ \hat{u}_{k+1}^{[k]} \end{array} \right\}, \]  

(11)

where matrices \( \mathcal{R}_k^T K_{k,k} \mathcal{R}_k \) and \( \Theta = \mathcal{R}_{k+1}^T K_{k+1,k+1} \mathcal{R}_{k+1} \) are symmetric while matrix \( \mathcal{R}_k^T K_{k,k} \mathcal{R}_{k+1} \) is not. These three matrices form the basis of the present development and can be obtained from any commercial finite element package.

Because nodal displacement and force components are dual variables, nodal forces are found as

\[ \hat{P}_k^{[k]} = -\frac{\partial V}{\partial \hat{u}_k^{[k]}} = -\Gamma \hat{u}_k - \Psi \hat{u}_{k+1}, \]

(12a)

\[ \hat{P}_{k+1}^{[k+1]} = -\frac{\partial V}{\partial \hat{u}_{k+1}^{[k+1]}} = \Psi^T \hat{u}_k + \Theta \hat{u}_{k+1}, \]  

(12b)
where a minus sign was introduced in the definition of \( \hat{P}_k \) to ensure consistent sign conventions with the FEM analysis. Equations (12) can be combined into the following form

\[
\hat{X}_{k+1} = S \hat{X}_k,
\]

(13)

where arrays \( \hat{X}_k^T = \{ \hat{u}_k, \hat{P}_k \} \) and \( \hat{X}_{k+1}^T = \{ \hat{u}_{k+1}, \hat{P}_{k+1} \} \) store the nodal displacements and forces at cross-sections \( s_k \) and \( s_{k+1} \), respectively, and matrix \( S \), of size \( 2n \times 2n \), is defined as

\[
S = \begin{bmatrix} -\Psi^{-1} & \Psi^{-1} \\ \Psi T - \Theta \Psi^{-1} & -\Theta \Psi^{-1} \end{bmatrix}.
\]

(14)

Because \( S^T J S = J \), where matrix \( J \) is defined in appendix A, matrix \( S \) is symplectic, see eq. (53). Equations (13) express transfer relationships between state variables \( X_k \) and \( X_{k+1} \) and hence, matrix \( S \) is referred to as the “symplectic transition matrix” or “transfer matrix.”

4 Central solutions

For a span-wise uniform beam with a helicoidal reference line, Han and Bauchau [20] proved that Saint-Venant’s solutions exist because rigid-body motions create no strains. This section proves that this fact remains true for periodic beams with unit cells of arbitrary configuration sharing identical end cross-sections. The rigid-body motion of a unit cell is investigated in section 4.1. The construction of the subspace associated with the unit eigenvalues of the symplectic transfer matrix is discussed in section 4.2. The duality between the rigid-body motions and the existence of Saint-Venant’s solutions is discussed in 4.3, leading to a formal proof for the solvability of the problem. The projection matrix is determined in section 4.4. Finally, the central solutions and equivalent sectional stiffness are obtained in section 4.5.

4.1 Rigid-body motion

Consider a unit cell undergoing rigid-body motion \( U \) described in section 2.3. The nodal displacements at end sections \( s_k \) and \( s_{k+1} \) are \( \hat{u}_k = ZU^k \) and \( \hat{u}_{k+1} = ZU^{k+1} \), respectively. Because rigid-body motions create no strain, the corresponding nodal forces must vanish, \( \hat{P}_k = \hat{P}_{k+1} = 0 \). Introducing \( X_k^T = \{(ZU^k)^T, 0^T\} \) and \( X_{k+1}^T = \{(ZU^{k+1})^T, 0^T\} \) into the governing equations (13) and applying the change of frame operation (5) leads to

\[
S \hat{Z} = \hat{Z} R^{-1},
\]

(15)

where matrix \( \hat{Z}^T = \{ Z^T, 0 \} \) is of size \( 2n \times 6 \). Introducing transfer matrix (14) then yields two matrix identities,

\[
\Gamma \hat{Z} = -\Psi \hat{Z} R^{-1}, \quad \Psi^T \hat{Z} = -\Theta \hat{Z} R^{-1}.
\]

(16a, 16b)

Manipulation of these identities leads to

\[
M \hat{Y} R^{-1} R^{-1} + G \hat{Y} R^{-1} + E \hat{Y} = 0,
\]

(17)

where the following matrices, each of size \( (n+6) \times (n+6) \), were defined

\[
M = \begin{bmatrix} \Psi^T & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} \Gamma + \Theta & \Psi Z \\ Z^T \Theta & 0 \end{bmatrix}, \quad E = \begin{bmatrix} \Psi^T & \Theta Z \\ Z^T & \Theta \end{bmatrix}.
\]

(18)
Matrix $Y$, of size $6 \times 6$, is

$$Y = \begin{bmatrix} \frac{Z}{\alpha} - R & \frac{Z}{\alpha} \end{bmatrix},$$

(19)

where $\alpha$ is an arbitrary matrix of size $6 \times 6$.

Identity (17) expresses the fact that rigid-body motions create no strain and can be recast as

$$\mathbb{S} \tilde{Y} = 0,$$

(20)

where notation $\tilde{\cdot}$ indicates the array obtained by stacking the columns of matrix $(\cdot)$ into a single array. Matrix $\mathbb{S}$, of size $(6n + 36) \times (6n + 36)$, is defined as

$$\mathbb{S} = (R^{-T}R^{-T}) \otimes M^T + R^{-T} \otimes G^T + I_6 \otimes E^T,$$

(21)

where symbol $\otimes$ indicates the Kronecker product of two matrices. Equation (20) implies that vector $\tilde{Y}$ belongs to the null space (kernel) of matrix $\mathbb{S}$, i.e., $\tilde{Y} \in \ker(\mathbb{S})$. Because matrix $\alpha$ is arbitrary, $\dim(\ker(\mathbb{S})) \geq 36$. Numerical experimentation shows that the equality holds.

### 4.2 Subspace associated with unit eigenvalues

Let the columns of matrix $U^{(c)}$, of size $2n \times 12$, span the invariant subspace of matrix $\mathbb{S}$ associated with its unit eigenvalues, which implies

$$\mathbb{S}U^{(c)} = U^{(c)} \hat{S}^{(c)},$$

(22)

where matrix $\hat{S}^{(c)}$ is of size $12 \times 12$. Without loss of generality, symplectic orthogonality of the columns of matrix $U^{(c)}$ can be imposed, i.e., matrix $U^{(c)}$ is symplectic, see appendix B. Because the product of symplectic matrices is symplectic, the fact that matrix $\mathbb{S}$ is symplectic implies that reduced matrix $\hat{S}^{(c)}$ is also symplectic. Equation (22) can be viewed as a projection of the problem onto the subspace of matrix $\mathbb{S}$ associated with its unit eigenvalues; matrix $U^{(c)}$ is the projection matrix.

The structure of the projection matrix is selected as follows,

$$U^{(c)} = \begin{bmatrix} Z & W \\ 0 & Y \end{bmatrix} = \begin{bmatrix} Z & W \end{bmatrix}.$$

(23)

The first six columns are associated with the rigid-body motions and echo eqs. (15); the last six columns span Saint-Venant’s solutions and matrices $W$ and $Y$, both of size $n \times 6$, are yet unknown. Because matrix $U^{(c)}$ is symplectic, the following conditions must be satisfied,

$$Z^TY = I, \quad W^TY = 0.$$

(24a)

(24b)

Identity (15) implies that the top- and bottom-left $6 \times 6$ sub-matrices of $\hat{S}^{(c)}$ are $R^{-1}$ and $0$, respectively. Because matrix $\hat{S}^{(c)}$ is symplectic, its structure must then be as follows,

$$\hat{S}^{(c)} = \begin{bmatrix} R^{-1} & B \\ 0 & R^T \end{bmatrix}.$$

(25)

The upper-right partition, denoted $B$, must satisfy identity $(RB) = (RB)^T$. Matrix $RB$ will be interpreted later as the reduced $6 \times 6$ compliance matrix of the unit cell.
4.3 Solvability conditions

Projection (22) consists of two sets of equations. The first set, $S\hat{Z} = \hat{Z}R^{-1}$, is identical to eqs. (15). Multiplying the second set, $SW = \hat{Z}B + WR^T$, by $S^T\hat{J}$ leads to the governing equations of the problem $JWR^{-T} = S^T\hat{J}(\hat{Z}BR^{-T} + W)$. Introducing symplectic matrix (14) yields two matrix equations,

$$\begin{align*}
\Theta^T W + \Theta^T \hat{Z} \hat{B} + \Psi^T WR^{-T} &= Y, \quad (26a) \\
\Psi^T W + \Psi^T \hat{Z} \hat{B} + \Gamma WR^{-T} &= -Y \hat{R}^{-T}, \quad (26b)
\end{align*}$$

where $\hat{B} = BR^{-T}$. First, pre-multiplying eq. (26a) by $Z^T$ and using condition (24a) yields the following 6 × 6 equations for matrices $W$ and $\hat{B}$

$$Z^T\Theta^T W + Z^T\Theta^T \hat{Z} \hat{B} + Z^T\Psi^T WR^{-T} = I. \quad (27)$$

Next, matrix $Y$ is expressed in terms of $W$ and $\hat{B}$ by introducing (26a) into eqs. (26b) leading to the other $n \times 6$ equations for matrices $W$ and $\hat{B}$. Finally, combining these $(n + 6) \times 6$ equations yields

$$M^T \hat{X} R^{-T} R^{-T} + C^T \hat{X} R^{-T} + E^T \hat{X} = T. \quad (28)$$

where $T = [0^T_{n \times 6} I_6]$. Linear system (28) features $(n + 6) \times 6$ unknowns, $\hat{X}^T = [W^T \hat{B}^T]$, and matrices $M$, $C$, and $E$ are defined by eq. (18). It can be recast in a compact form as

$$\mathbb{S}^T(\hat{X}) = \mathbb{T}, \quad (29)$$

where matrix $\mathbb{S}$ is defined by eq. (21). Matrix $\mathbb{S}^T$ is singular and identities (16) yield its null space as

$$Y = \begin{bmatrix} Z \\ -\alpha + R^{-1} \alpha \end{bmatrix}, \quad (30)$$

where $\alpha$ is an arbitrary matrix of size 6×6, which implies dim(\ker($\mathbb{S}^T$)) ≥ 36. Numerical experimentation shows that the equality holds.

Because matrix $\mathbb{S}^T$ is singular, solutions of linear system (29) exist if and only if the solvability conditions are satisfied. In view of eqs. (20), the solvability conditions are $\mathbb{V}^T(\mathbb{T}) = 0$ and are always satisfied because $(\mathbb{V})^T(\mathbb{T}) = \text{tr}(\mathbb{R}^{-1} \alpha + \alpha \mathbb{R}^{-1}) = -\text{tr}(\mathbb{R}^{-1} \alpha) + \text{tr}(\alpha \mathbb{R}^{-1}) = 0$. This discussion leads to a fundamental theoretical result: *Saint-Venant’s solutions exist because rigid-body motions create no strains.*

4.4 Determination of the projection matrix

This section outlines the procedure for the determination of the projection matrices. For straight beams, linear system (28) splits into two subsystems, which can be solved recursively

$$\begin{align*}
(E + C + M) [X_4, X_5, X_6] &= [T_4, T_5, T_6], \quad (31a) \\
(E + C + M) [X_1, X_2, X_3] &= [T_1, T_2, T_3] - (2M + C) [X_4, X_5, X_6] T_i^{[k]}, \quad (31b)
\end{align*}$$

where $X_i$ and $T_i$ denote the $i^{th}$ column of matrices $X$ and $T$, respectively.

For curved and twisted beams, introducing transformation $\hat{R} = \hat{C}^{-1} R C$ into eq. (28) and right multiplying $\hat{S}^{-T}$, leads to

$$M \hat{X}^o \hat{R}^{-T} \hat{R}^{-T} + C \hat{X}^o \hat{R}^{-T} - E \hat{X}^o = T^o, \quad (32)$$

where $\hat{X}^o$ denotes the $o^{th}$ column of $\hat{X}$. Numerical experimentation shows that the equality holds.
where \( \mathbf{X}^b = \mathbf{X} \mathbf{C}^{-T} \) and \( \mathbf{Z}^b = \mathbf{Z} \mathbf{C}^{-T} \). Equation (32) breaks down into three uncoupled problems, which can be solved recursively,

\[
\begin{align*}
\left( \mathbf{E} + \mathbf{G} + \mathbf{M} \right) [\mathbf{X}_1, \mathbf{X}] &= [\mathbf{I}_1, \mathbf{I}], \\
\left( \mathbf{B} - j \mathbf{B} \right) (\mathbf{X}^b + j \mathbf{X}^b) &= (\mathbf{I}_b + j \mathbf{I}_b), \\
\left( \mathbf{B} - j \mathbf{B} \right) (\mathbf{X}^b + j \mathbf{X}^b) &= (\mathbf{I}_b + j \mathbf{I}_b) + (\mathbf{A} + j \mathbf{A}) (\mathbf{X}^b + j \mathbf{X}^b),
\end{align*}
\]

(33a) \( 33b \) \( 33c \)

where \( j = \sqrt{-1}, \mathbf{B} = \mathbf{E} + \cos \phi \mathbf{G} + \cos 2\phi \mathbf{M}, \mathbf{B} = \sin \phi \mathbf{G} + \sin 2\phi \mathbf{M}, \mathbf{A} = d \cos \phi \mathbf{G} + 2d \cos 2\phi \mathbf{M} \).

Because system (28) is 36 times singular, its general solutions take the following form,

\[
\begin{align*}
\mathbf{W} &= \mathbf{W}^+ + \mathbf{Z} \gamma, \\
\mathbf{B} &= \mathbf{B}^+ - \gamma + \mathbf{R}^{-1} \gamma \mathbf{R}^{-T},
\end{align*}
\]

(34a) \( 34b \)

where notation \( (\cdot)^+ \) denotes a particular solution of the problem and coefficients \( \gamma \) are found easily as \( \gamma = -\mathbf{Y}^T \mathbf{W}^+ \) because the solution must satisfy condition (24b). Finally, \( \mathbf{B} = \mathbf{B} \mathbf{R}^T \).

### 4.5 Central solutions

The two previous sections show that projection matrix \( \mathbf{U}^{(e)} \) always exists because it is the solution of a singular, but solvable linear system. Next, the central solutions are obtained from a coordinate transformation of the following type

\[
\begin{align*}
\hat{\mathbf{X}}^{(c)} &= \mathbf{U}^{(e)} \mathbf{L}^{(c)}, \\
\hat{\mathbf{X}}^{(c)} &= \mathbf{U}^{(e)} \mathbf{L}^{(c+1)},
\end{align*}
\]

(35a) \( 35b \)

where arrays \( \hat{\mathbf{X}}^{(c)} \) and \( \hat{\mathbf{X}}^{(c)} \) store the nodal displacements and forces associated with the central solution at cross-section \( \mathbf{s}_k \) and \( \mathbf{s}_{k+1} \), respectively, and generalized coordinates \( \mathbf{L}^{(c)} = \{ \mathbf{U}^{(c)} \mathbf{F}^{(c)} \} \) and \( \mathbf{L}^{(c+1)} = \{ \mathbf{U}^{(c+1)} \mathbf{F}^{(c+1)} \} \) store the components of average sectional displacements and sectional stress resultants for cross-sections \( \mathbf{s}_k \) and \( \mathbf{s}_{k+1} \), respectively, resolved in frames \( \mathbf{F}_k \) and \( \mathbf{F}_{k+1} \), respectively.

Introducing coordinate transformation (35) into the governing equations (13) yields \( \mathbf{U}^{(e)} \mathbf{L}^{(c)} = \mathbf{S} \mathbf{U}^{(e)} \mathbf{L}^{(c)} \) and pre-multiplication by \( \mathbf{U}^{(c)} \mathbf{F} \) provides the reduced governing equations as \( \mathbf{L}^{(c)} = \mathbf{S} \mathbf{L}^{(c)} \). More specifically,

\[
\begin{bmatrix}
\mathbf{U}^{(k+1)} \\
\mathbf{F}^{(k+1)}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{R}^{-1} & \mathbf{B} \\
0 & \mathbf{R}^{-1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}^{(k)} \\
\mathbf{F}^{(k)}
\end{bmatrix},
\]

(36)

and the frame change relationships (5) and (6) then yield

\[
\begin{align*}
-\mathbf{F}^{(k)} &= (\mathbf{R} \mathbf{B})^{-1} (\mathbf{U}^{(k)} - \mathbf{U}^{(k+1)}), \\
\mathbf{F}^{(k+1)} &= (\mathbf{B} \mathbf{R}^{-T})^{-1} (-\mathbf{U}^{(k+1)} + \mathbf{U}^{(k+1)}).
\end{align*}
\]

(37a) \( 37b \)

where the minus sign was introduced in (37a) to make the sign convention consistent with those of the finite element analysis. Clearly, matrices \( \mathbf{R} \mathbf{B} \) and \( \mathbf{B} \mathbf{R}^{-T} \) store the components of the unit cell’s compliance matrix, resolved in frames \( \mathbf{F}_k \) and \( \mathbf{F}_{k+1} \), respectively.
4.6 Equivalent beam model

For uniform beams, Han and Bauchau [19] have shown that \( S^c = \exp(L_c H^c) \), which echoes eq. (9). Of course, for uniform beams, \( L_c \) can be chosen arbitrarily. For a slice of the beam, the Hamiltonian matrix is

\[
\hat{H}^c = \begin{bmatrix}
\tilde{K}^* & S^* \\
0 & \tilde{K}^T
\end{bmatrix},
\]

(38)

where \( S^* \) is the beam’s sectional compliance matrix and \( \tilde{K}^* \) the generalized curvature tensor, both resolved in the material basis. The explicit form of the latter is

\[
\tilde{K}^* = \begin{bmatrix}
k^* & t^* \\
0 & \tilde{k}^*
\end{bmatrix},
\]

(39)

where \( k^* \) and \( t^* \) stores the components of curvature and tangent vectors, respectively, of the beam’s reference line and notation \((\cdot)\) denotes the antisymmetric matrix formed with the components of vector \((\cdot)\). For uniform beams, the Hamiltonian matrix is obtained from a two-dimensional analysis of the cross-section and the transfer matrix for any point along the beam is then evaluated as

\[
\hat{S}^c(x_1) = \exp(x_1 \hat{H}^c).
\]

In contrast, the opposite sequence is followed for periodic beams: as outlined in section 3.2, the transfer matrix is obtained first from a three-dimensional analysis of a unit cell of length \( L_c \) using a general purpose finite element code and the Hamiltonian matrix is then evaluated as \( \hat{H}^c = \log(\hat{S}^c)/L_c \). With the Hamiltonian matrix at hand, it becomes possible to define an equivalent, uniform beam model. The curvature and tangent vectors of the reference line of the equivalent beam are extracted from the generalized curvature tensor, \( \tilde{K}^* \), through eq. (39) and the uniform compliance of the equivalent beam is extracted from the upper-right partition of the Hamiltonian matrix, see eq. (38).

For straight beams, methods for predicting sectional stiffness based on three-dimensional finite element analysis have appeared in the literature. Ghiringhelli et al. [26] proposed an approach based on the sectional analysis method developed by Giavotto et al. [27] and Borri and Merlini [28]. Fatmi and Zenzri [29] proposed a similar approach based on the work of Ladèvèze and Simmonds [30, 31]. In both cases, equivalent beam properties were obtained from an energy minimization argument; the connection between the transfer matrix and the Hamiltonian matrix was not identified.

5 Extremity solutions

Symplectic transfer matrix \( \hat{S} \) presents \( n - 6 \) reciprocal pairs of non-unit eigenvalues, denoted as \( \mu_i (\|\mu_i\| \leq 1) \) and \( 1/\mu_i \), \( i = 1, 2, \ldots, n - 6 \), which are associated with the extremity solutions

\[
S U^{(e)} = U^{(e)} S^{(e)}.
\]

(40)

Matrices \( S^{(e)} \) and \( U^{(e)} \), which store the non-unit eigenvalues and associated eigenvectors, respectively, are defined as

\[
S^{(e)} = \begin{bmatrix}
\Lambda & 0 \\
0 & \Lambda^{-1}
\end{bmatrix}, \quad U^{(e)} = \begin{bmatrix}
U_a & U_b \\
V_a & V_b
\end{bmatrix},
\]

(41)

where \( \Lambda = \text{diag}(\mu_i) \) and subscripts \( a \) and \( b \) indicate the eigenvectors associated with eigenvalues of magnitude larger smaller and larger than one, respectively.
The extremity solutions are obtained from the following coordinate transformation,

\[ \hat{X}^{(e)}_{k+1} = \mathbf{U}^{(e)} \hat{L}^{(e)}_{k+1}, \]  
\[ \hat{X}^{(e)}_{k} = \mathbf{U}^{(e)} \hat{L}^{(e)}_{k}, \]  

(42a)  
(42b)

where arrays \( \hat{X}^{(e)}_{k+1} \) and \( \hat{X}^{(e)}_{k} \) store the nodal displacements and forces associated with the extremity solutions at sections \( s_{k+1} \) and \( s_k \), respectively, and generalized coordinates \( \hat{L}^{(e)}_{k+1} = \{ \hat{a}^{T}_{k+1}, \hat{b}^{T}_{k+1} \} \) and \( \hat{L}^{(e)}_{k} = \{ \hat{a}^{T}_{k}, \hat{b}^{T}_{k} \} \) store the corresponding magnitudes of excitation of each mode.

Introducing coordinate transformation (42) into the governing equations (13) yields \( \mathbf{U}^{(e)} \hat{L}^{(e)}_{k+1} = \mathbf{S} \mathbf{U}^{(e)} \hat{L}^{(e)}_{k} \). Finally, pre-multiplication by \( \mathbf{U}^{(e)} \hat{L}^{(e)}_{k+1} \) provides the reduced governing equations as \( \mathbf{L}^{(e)}_{k+1} = \mathbf{S}^{(e)} \mathbf{L}^{(e)}_{k} \), and more specifically,

\[ \begin{align*}
\hat{a}^{T}_{k+1} & = \Lambda \hat{a}^{T}_{k}, \\
\hat{b}^{T}_{k+1} & = \Lambda^{-1} \hat{b}^{T}_{k}.
\end{align*} \]  
\[ (43a) \]  
\[ (43b) \]

For uniform beams, Han and Bauchau [19] have shown that \( \mathbf{S}^{(e)} = \exp(\mathbf{L}_{e} \mathbf{H}^{(e)}) \), which echoes eq. (9). Of course, for uniform beams, \( \mathbf{L}_{e} \) can be chosen arbitrarily. For a slice of the beam, the Hamiltonian matrix is diagonal, \( \mathbf{H}^{(e)} = \left[ \text{diag}(\lambda_i), -\text{diag}(\lambda_i) \right] \), and hence, \( \mathbf{S}^{(e)} = \left[ \text{diag}(\mu_i), \text{diag}(1/\mu_i) \right] \), where \( \mu_i = \exp(\mathbf{L}_{e} \lambda_i) \).

6 Solutions of periodically heterogenous beams

Combination of the central solutions (36) and extremity solutions (43) leads to the complete solutions of periodically heterogenous beams. The associated boundary conditions are presented in section 6.2. Finally, the local three-dimensional stress fields are recovered in section 6.3.

6.1 Complete solutions

Let the beam consist of \( m \) unit cells, and hence, its end sections are \( s_1 \) and \( s_{m+1} \). The nodal displacements and forces at section \( i \) are stored in array \( \hat{X}^{i} \), \( i = 1, 2, 3, \ldots, m + 1 \). The transfer matrix for the entire beam is then \( \hat{X}^{(e)}_{m+1} = \mathbf{S} \hat{X}^{(e)}_{m-1} = \cdots = \mathbf{S}^{m} \hat{X}^{(e)}_{1} \). Introducing projections (35) and (42), the generalized coordinates at section \( \ell \), \( \ell = 1, 2, 3, \ldots, m + 1 \), denoted \( \mathbf{L}^{(e)}_{\ell} = \{ \mathbf{L}^{(e)}_{\ell}, \hat{a}^{T}_{\ell}, \hat{b}^{T}_{\ell} \} \), can be expressed as

\[ \mathbf{L}^{(e)}_{\ell} = \mathbf{G}^{(e)} \mathbf{\tilde{L}}, \]  
\[ (44) \]

where \( \mathbf{\tilde{L}}^{T} = \{ \mathbf{L}^{(e)}_{\ell}, \hat{a}^{T}_{\ell}, \hat{b}^{T}_{m+1} \} \) and the transfer matrix is \( \mathbf{G}^{(e)} = \text{diag}(\mathbf{S}^{(e)}_{\ell}, \Lambda^{\ell-1}, \Lambda^{m+1-\ell}) \). The explicit expression of \( \mathbf{S}^{(e)}_{\ell} \) is

\[ \mathbf{S}^{(e)}_{\ell} = \left[ \begin{array}{cc} \mathbf{R}^{\ell} & \mathbf{B} \\
0 & \mathbf{R}^{-\ell} \end{array} \right], \]  
\[ (45) \]

where \( \mathbf{B} = \sum_{i=0}^{\ell} \mathbf{R}^{-i} \mathbf{B} \mathbf{R}^{(\ell-i)} \).

Once the generalized coordinates are determined, the combination of the central solutions (36) and extremity solutions (43) leads to the complete solution

\[ \hat{X}_{\ell} = \left[ \begin{array}{c} \mathbf{U}^{(e)} \\
\mathbf{U}^{(e)} \mathbf{G}_{\nu} \end{array} \right] \mathbf{G}^{(e)} \mathbf{\tilde{L}} = \left[ \begin{array}{c} \mathbf{Q} \\
\mathbf{P} \end{array} \right] \mathbf{G}_{\nu} \mathbf{\tilde{L}}, \]  
\[ (46) \]

where matrices \( \mathbf{Q} = [\mathbf{Z}, \mathbf{W}, \mathbf{U}, \mathbf{U}^{(e)}] \) and \( \mathbf{P} = [0, \mathbf{Y}, \mathbf{V}, \mathbf{V}^{(e)}] \). The only remaining unknowns are generalized coordinates \( \mathbf{\tilde{L}} \), which will be evaluated by the boundary conditions.
6.2 Boundary conditions and boundary effect

Boundary conditions [32, 33] are enforced in the weak form,

\[
\left[ \delta \hat{u}^T_{m+1}(\hat{P}_{m+1} - \hat{P}_{m+1}) - \delta \hat{P}^T_{m+1}(\hat{u}_{m+1} - \hat{u}_{m+1}) \right] - \left[ \delta \hat{u}^T_1(\hat{P}_1 - \hat{P}_1) - \delta \hat{P}^T_1(\hat{u}_1 - \hat{u}_1) \right] = 0, \tag{47}
\]

where subscript 1 and \( m + 1 \) indicate nodal displacements and forces at the beam’s end sections \( s_1 \) and \( s_{m+1} \), respectively, and notation \( \delta \) indicates prescribed nodal displacements or forces at the beam’s end sections. Three types of boundary conditions are commonly encountered: \( (1) \) nodal displacements are prescribed over the end sections, \( (2) \) nodal forces are prescribed over the end sections, or \( (3) \) nodal displacements and forces are prescribed over mutually exclusive portions of the end sections. For simplicity, it is assumed that nodal displacements are specified at the beam’s end sections, \( \hat{u}_1 = \hat{u}_1 \) and \( \hat{u}_{m+1} = \hat{u}_{m+1} \). Other types of boundary conditions can be derived as well. Introducing the specified values into eq. (47), and noticing that the variations \( \delta \hat{u}_1 \) and \( \delta \hat{u}_{m+1} \) vanish yields

\[
\delta \hat{P}^T_{m+1} [\hat{u}_{m+1} - \hat{u}_{m+1}] - \delta \hat{P}^T_1 [\hat{u}_1 - \hat{u}_1] = 0. \tag{48}
\]

Introducing closed form solutions (46) into eq. (48), yield

\[
\delta \mathbf{L}^T \left( \mathbf{A} \mathbf{L} - \mathbf{b} \right) = 0, \tag{49}
\]

where matrix \( \mathbf{A} = \mathbf{Q}^T \mathbf{G}^T \mathbf{G}^T \mathbf{Q} \) and array \( \mathbf{b} = \mathbf{Q}^T \mathbf{G}^T \mathbf{P}^T \hat{u}_{m+1} - \mathbf{Q}^T \mathbf{G}^T \hat{u}_1 \). Clearly, the enforcement of boundary conditions (49) leads to a set of equations of unknowns, \( \hat{u}_k \) and the corresponding solutions can be found easily.

As implied by eq. (49), the central and extremity solutions interact at the boundaries. The magnitudes of the extremity modes alter those of the central solutions through the coupling implied by matrix \( \mathbf{Q}^T \mathbf{G} \). For a specific problem, some extremity solutions may play a significant role because of strong coupling at the boundaries although they do not propagate far into the beam.

6.3 Stress recovery

Combining eqs. (36) and (43) yields the nodal displacement field at section \( s_k \),

\[
\hat{u}_k = \mathbf{Z}_k \mathbf{U}_k^{[k]} + \mathbf{W}_k \mathbf{P}_k^{[k]} + \mathbf{U}_k \mathbf{a}_k + \mathbf{U}_k \mathbf{b}_k. \tag{50}
\]

The first two terms on the right-hand side describe the contributions of the rigid-section motion and stress resultants, respectively, the last two terms describe the contributions of the extremity solutions associated with reciprocal non-unit eigenvalues. In eq. (50), the last three terms describe the effect of warping. Finally, the stress field within each unit cells can be recovered by imposing the nodal displacement field over each cell’s end sections to the finite element model of the unit cell.

7 Numerical examples

To validate the proposed solution procedure for periodically heterogenous beam problem, three numerical examples will be presented. In each case, the following procedure is followed: \( (1) \) perform the three-dimensional analysis of a unit cell and obtain the associated stiffness matrices \( \mathbf{K}_{k,k} \), \( \mathbf{K}_{k,k+1} \) and \( \mathbf{K}_{k+1,k+1} \); \( (2) \) determine the central solutions and the equivalent sectional stiffness matrix \( \mathbf{K}_m^{s-1} \), as discussed in section 4; \( (3) \) determine the extremity solutions and the associated non-unit eigenvalues, as discussed in section 5; \( (4) \) determine the generalized coordinates by enforcing the corresponding boundary conditions (49), \( (5) \) recover stress field in the unit cells, as discussed in section 6.3. To validate the predictions, the associated full beam model are also analysed using ABAQUS. In the examples, the units of stiffness matrix components \( S_{ij}^{s-1} \) are as follows: N·m\(^{-1}\) for \( i, j = 1, 2, 3; \) N·m for \( i, j = 4, 5, 6; \) and N for all other components.
7.1 Helicoidal beam

The three-dimensional helicoidal beam shown in fig. 4 has a uniform rectangular cross-section of size $b \times h = 0.1 \times 0.06$ m$^2$ and its reference line, which passes through the cross-section’s lower-inner corner, is a helix of equation $x_1 = \rho \cos \theta$, $x_2 = \rho \sin \theta$, $x_3 = p$, where $\rho = 0.35$ m is the radius and $p = 0.1$ m/rad the pitch. Unit vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$ are the unit tangent, normal, and binormal vectors, respectively, of the helicoidal reference line. The beam’s cross-section lies in the plane defined by $\vec{b}_2$ and $\vec{b}_3$.

![Figure 4: Configuration and meshes of a slice of helicoidal beam.](image)

Although this beam is uniform along its span, it was analysed as a periodic structure with a unit-cell length of $L_c = 0.0351241$ m. The unit-cell analysis was performed using ABAQUS with a mesh of $5 \times 3 \times 5$ 20-node brick elements. Using the proposed approach, the beam’s stiffness matrix was predicted to be

$$
\frac{S^{*-1}}{10^6} = \begin{bmatrix}
366 & -7.30 & -11.2 & 0.955 & 11.0 & 16.9 \\
-7.3 & 117. & -0.350 & -3.51 & -0.220 & -0.351 \\
-11.2 & -0.350 & 113 & -5.16 & -0.322 & -0.660 \\
0.955 & -3.51 & -5.16 & 0.449 & 0.0282 & 0.0667 \\
11.0 & -0.220 & -0.322 & 0.0282 & 0.438 & 0.506 \\
16.6 & -0.351 & -0.660 & 0.0667 & 0.506 & 1.07
\end{bmatrix}.
$$

Due to the curvature of the beam and the choice of the location of the reference axis, this stiffness matrix is fully populated, as expected.

To validate these predictions, the Hamiltonian analysis [20] was also performed, using a two-dimensional mesh of $5 \times 3$ 8-node elements over the cross-section, leading to the following stiffness matrix

$$
\frac{S^{*-1}}{10^6} = \begin{bmatrix}
365 & -6.71 & -11.0 & 0.926 & 11.0 & 16.8 \\
-6.71 & 117. & -0.294 & -3.50 & -0.202 & -0.323 \\
-11.0 & -0.294 & 112. & -5.16 & -0.315 & -0.648 \\
0.926 & -3.50 & -5.16 & 0.449 & 0.0274 & 0.0653 \\
11.0 & -0.202 & -0.315 & 0.0274 & 0.437 & 0.504 \\
16.8 & -0.323 & -0.648 & 0.0653 & 0.504 & 1.07
\end{bmatrix}.
$$

Table 1 shows the lowest nine pairs of eigenvalues presenting non-vanishing real parts. The small discrepancies come from mismatched meshes at the slice’s two end sections: due to the beam’s curvature, the mesh generation process does not yield identical meshes at the two end cross-sections.

7.2 Pre-twisted ribbed box beam

The pre-twisted, ribbed beam shown in fig. 5 is of length $L = 1.0$ m and has a uniform rectangular box cross-section of size $b \times h = 0.1 \times 0.06$ m$^2$. Ten ribs are located at stations $\alpha_1 = 0.05, 0.15, 0.25, \cdots, 0.95$ m resulting in a unit-cell length of $L_c = 0.1$ m. The beam is pre-twisted at a rate of
Table 1: Eigenvalues with non-vanishing real parts, m⁻¹.

<table>
<thead>
<tr>
<th>Present ± log(µᵢ)</th>
<th>Hamiltonian ±λᵢ</th>
</tr>
</thead>
<tbody>
<tr>
<td>±50.11</td>
<td>±50.09</td>
</tr>
<tr>
<td>±50.53 ± 12.50 j</td>
<td>±50.52 ± 12.52 j</td>
</tr>
<tr>
<td>±50.41 ± 28.01 j</td>
<td>±50.49 ± 28.02 j</td>
</tr>
<tr>
<td>±77.83 ± 23.01 j</td>
<td>±77.82 ± 23.00 j</td>
</tr>
<tr>
<td>±78.51 ± 39.14 j</td>
<td>±79.08 ± 39.31 j</td>
</tr>
</tbody>
</table>

2π/(5Lc) rad/m. Both the box and ribs are of thickness t = 0.012 m and are made of an isotropic material with Young’s modulus E = 73 GPa and Poisson’s ratio ν = 0.3.

The beam is clamped at the root while the cross-section at the other end is constrained to remain rigid. The tip motion is prescribed as follows: point B, located at the geometric center of the cross-section, translates along unit vector ȓ₃ (u₃ = 0.1 mm) and the plane of the section rotates about unit vector ȓ₁ (θ₁ = 0.1 mrad). The unit cells are labeled 1, 2, · · ·, 10, starting from the clamped end. For the present approach, a unit cell was analysed. The finite element analysis of the unit cell was performed using ABAQUS with a mesh of 1,144 4-node quadrilateral S8R6 shell elements.

The beam’s equivalent sectional stiffness matrices predicted by the proposed approach is

\[
\begin{bmatrix}
14.0 & 1.43 & 0.175 & -0.0385 & -0.129 \\
0.175 & 2.20 & 0.0087 & -0.0385 & -0.129 \\
-0.0385 & 0.0087 & 0.0935 & 0.0203
\end{bmatrix}
\]

Figure 5: Configuration and mesh of a pre-twisted, ribbed box beam.

Figures 6, 7, and 8 show the distributions of the axial stress flow (Units: [N·m⁻¹]) over a portion of the 1th cell predicted by the present approach with an increasing number of extremity modes, nₑ = 0, 4, 56 and 59, respectively.

Note the presence of extension/twist and shear/curvature coupling terms arising from the pre-twist of the structure. The cell’s first twelve pairs of non-unit eigenvalues are listed in table 1. Within a cell’s length, all extremity modes decay to less than 1% except for the first 6 pairs and to 1%₀₀₀ except for the first 18 pairs. Clearly, the decay length of extremity modes is shorter than the unit-cell length but although they decay rapidly, higher-order modes may affect the solution through coupling with the central solution at the boundaries, as discussed in section 6.2.
Table 2: Non-unit eigenvalues for a unit cell of the ribbed box beam, m\(^{-1}\).

| \(\mu\) (||\(\mu\)|| \leq 1.0) | \(\mu^{-1}\) |
|-----------------|-----------------|
| \(3.35 \times 10^{-2} \pm 9.67 \times 10^{-2} j\) | \(3.20 \pm 9.23 j\) |
| \(-6.73 \times 10^{-3} \pm 4.58 \times 10^{-2} j\) | \(-3.14 \pm 2.14 \times 10^1 j\) |
| \(4.47 \times 10^{-3} \pm 1.37 \times 10^{-2} j\) | \(2.15 \times 10^1 \pm 6.60 \times 10^1 j\) |
| \(-7.82 \times 10^{-3} \pm 3.09 \times 10^{-3} j\) | \(-1.11 \times 10^2 \pm 4.37 \times 10^1 j\) |
| \(-5.09 \times 10^{-3}\) | \(-1.96 \times 10^2\) |
| \(3.15 \times 10^{-3}\) | \(3.17 \times 10^2\) |

Figure 6: Axial stress flow over the 1\(^{st}\) cell, proposed approach \((n_e = 0)\).

Figure 7: Axial stress flow over the 1\(^{st}\) cell, proposed approach \((n_e = 4)\).

To validate the predictions, a fully three-dimensional model of the beam was analyzed in ABAQUS using 11,440 4-node quadrilateral S8R6 shell and the corresponding predictions are shown in fig. 9. Due to the existence of clamped end constraints, higher-order extremity modes need to be included, although they decay rapidly. With the inclusion of the first 59 extremity modes, the present predictions are in excellent agreement with those of ABAQUS.

The distributions of axial stress flow over the 5\(^{th}\) cell are presented in figs. 10 and 11 for the present approach and full three-dimensional ABAQUS model, respectively. Because this cell is located at the beam’s mid-span, the effect of extremity modes is negligible; the central solution yields predictions that match those of ABAQUS closely.

Finally, figs. 12 to 14 show the axial flow distributions over the 10\(^{th}\) cell for an increasing number of extremity modes. For this end cell, the effect of extremity modes is more pronounced: higher-order extremity modes must to be included to yield accurate predictions, see fig. 15. With the \(n_e = 59\), the present predictions converge to those of ABAQUS.

7.3 Spatial frame structure

The configuration of the spatial truss under investigation is shown fig. 16. The truss consists of a sequence for square sections with vertices at points \(P_1, P_2, P_3,\) and \(P_4\) \((b = h = 0.2 \text{ m})\) with a single cross-bar from point \(P_2^k\) to \(P_4^k\). A material frame \(\mathcal{F}_k = [\mathbf{B}_k, \mathcal{B} = (\mathbf{b}_1^k, \mathbf{b}_2^k, \mathbf{b}_3^k)]\) is attached at the center of each square section. The position vector of point \(\mathbf{B}_{k+1}\) with respect to point \(\mathbf{B}_k^k\) is
r_k = \{0.3, 0.3, 0.3\}^T \text{ m, and relative rotation tensor of basis } B_{k+1} \text{ with respect to } B_k \text{ is}

\begin{align*}
R^k &= \begin{bmatrix}
0.8830 & -0.2115 & 0.4190 \\
0.3214 & 0.9230 & -0.2115 \\
-0.3420 & 0.3214 & 0.8830
\end{bmatrix}.
\end{align*}

Both quantities are resolved in basis } B^k. \text{ The 10 beams forming the end sections of the unit cell and the nine beams connecting these two sections have solid circular sections of radius } r = 0.005 \text{ m and } r = 0.01 \text{ m, respectively. All the beams are made of an isotropic material with Young’s modulus } E = 73 \text{ GPa and Poisson’s ratio } \nu = 0.3. \text{ The periodic heterogenous beam structure under investigation is composed of 10 unit cells, see fig. 17. The beam is clamped at one end and is subjected to a tip force, } P = 100 \text{ N, at the point } P_1^{10}, \text{ along unit vector } \hat{b}_3^{10}. \text{ A single cell was analysed in ABAQUS using a mesh of 56 3-node B32 beam elements. The equivalent beam model identified by the proposed approach features a helicoidal reference line of curvature vector } \kappa^* = \{0.0116, 0.0166, 0.0116\} \text{ m}^{-1} \text{ and tangent } \tau^* =
The cell’s first 16 pairs of non-unit eigenvalues are listed in table 3. Within a cell’s length, all extremity modes decay to less than $10^{-6}$ except for the first 6 pairs.

The axial forces acting in the beams of the 1st, 5th, and 10th cells were evaluated and the predicted values are listed in tables 4, 5 and 6, respectively. Predictions were obtained with an increasing number of extremity modes, $n_e = 0, 6, 16, 18,$ and $20$. To validate the predictions, a beam model of the complete structure was analyzed in ABAQUS using 517 3-node B32 beam elements and these predictions are also listed in the table. For the 5th cell, the central solutions

$$\frac{S^{\ast-1}}{10^6} = \begin{bmatrix}
6.78 & 5.38 & 5.96 & 0.0296 & -0.0489 & 0.0937 \\
5.38 & 5.74 & 4.50 & 0.157 & -0.0548 & -0.0118 \\
5.96 & 4.50 & 6.73 & 0.137 & -0.125 & 0.0662 \\
0.0296 & 0.157 & 0.137 & 0.0703 & -0.0339 & -0.0365 \\
-0.0489 & -0.0548 & -0.125 & -0.0339 & 0.0370 & 0.00725 \\
0.0937 & -0.0118 & 0.0662 & -0.0365 & 0.00725 & 0.0389 
\end{bmatrix}.$$
only agree with predictions of ABAQUS. For the 1st, and 10th cells, the predictions of the proposed approach converge to those of ABAQUS as $n_e = 18$.

8 Conclusions

This paper has presented a novel strategy to investigate periodically heterogeneous beams. The approach is based on symplectic transfer matrix of the unit cell and the construction the subspace associated with the unit and non-unit eigenvalues of the transfer matrix’s is the key step of the proposed procedure. Two types of solutions were identified: central solutions or Saint-Venant’s solutions, associated with the twelve unit eigenvalues, and extremity solutions, associated with the reciprocal non-unit eigenvalue pairs. Extremity solutions decays exponentially in the cell and for the numerical examples presented here, the decay lengths of many of the extremity modes are shorter than the length of the unit cell. The central and extremity solutions are coupled through the boundary conditions at the beam’s ends.

The close connection of the proposed approach with the Hamiltonian approach was underlined and is the basis for the evaluation of the equivalent beam model. The theoretical statement, Saint-Venant’s solutions exist because rigid-body motions create no strains, first proved for uniform beams, was extended here for structures consisting unit cells of arbitrary three-dimensional configuration connected by identical cross-sections.

Numerical examples were presented to demonstrate the capabilities of the proposed approach. Predictions were compared to solutions of the Hamiltonian approach or fully FEM analysis and were found to be in excellent agreement with these results. Since the present analysis requires a unit
Table 3: Non-unit eigenvalues of the symplectic transfer matrix for a unit cell of the spatial frame, m⁻¹.

| \( \mu (||\mu|| \leq 1.0) \) | \( \mu^{-1} \) |
|---|---|
| -7.43 \(10^{-2} \pm 4.19 \times 10^{-2} \) j | -10.2 \( \pm 5.76 \) j |
| -4.59 \(10^{-2} \pm 7.11 \times 10^{-2} \) j | -6.41 \( \pm 9.93 \) j |
| -7.81 \(10^{-2} \pm 4.511 \times 10^{-3} \) j | -12.8 \( \pm 0.738 \) j |
| 6.79 \(10^{-2} \) | 14.7 |
| 5.63 \(10^{-2} \) | 17.8 |
| 4.93 \(10^{-2} \pm 2.28 \times 10^{-2} \) j | 16.7 \( \pm 7.72 \) j |
| -3.88 \(10^{-2} \pm 2.56 \times 10^{-2} \) j | -18.0 \( \pm 11.9 \) j |
| 4.12 \(10^{-2} \) | 24.2 |
| 1.81 \(10^{-2} \) | 55.4 |
| -2.22 \(10^{-2} \) | -4500 |
| 1.44 \(10^{-6} \) | 6.94 \(10^{5} \) |

Table 4: Axial forces of the beam components in the 1st cell, [N].

<table>
<thead>
<tr>
<th>Beam</th>
<th>ABAQUS</th>
<th>( n_e = 0 )</th>
<th>( n_e = 6 )</th>
<th>( n_e = 16 )</th>
<th>( n_e = 18 )</th>
<th>( n_e = 20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1^1 - P_4^1 )</td>
<td>101.5</td>
<td>293.6</td>
<td>292.5</td>
<td>-62.39</td>
<td>-101.5</td>
<td>-101.5</td>
</tr>
<tr>
<td>( P_1^2 - P_2^2 )</td>
<td>2836</td>
<td>-271.3</td>
<td>-271.2</td>
<td>-2847</td>
<td>-2836</td>
<td>-2836</td>
</tr>
<tr>
<td>( P_2^3 - P_3^3 )</td>
<td>544.6</td>
<td>982.9</td>
<td>977.6</td>
<td>530.1</td>
<td>544.5</td>
<td>544.5</td>
</tr>
<tr>
<td>( P_4^3 - P_2^1 )</td>
<td>-1420</td>
<td>-1316</td>
<td>-1309</td>
<td>-1422</td>
<td>-1420</td>
<td>-1420</td>
</tr>
<tr>
<td>( P_4^1 - P_4^2 )</td>
<td>1027</td>
<td>2486</td>
<td>2482</td>
<td>1057</td>
<td>1027</td>
<td>1027</td>
</tr>
<tr>
<td>( P_1^4 - P_4^4 )</td>
<td>337.7</td>
<td>-619.8</td>
<td>-607.8</td>
<td>306.8</td>
<td>337.7</td>
<td>337.7</td>
</tr>
<tr>
<td>( P_1^1 - P_2^1 )</td>
<td>1703</td>
<td>1347</td>
<td>1339</td>
<td>1719</td>
<td>1703</td>
<td>1703</td>
</tr>
<tr>
<td>( P_3^3 - P_3^1 )</td>
<td>-162.9</td>
<td>-655.7</td>
<td>-653.1</td>
<td>-125.3</td>
<td>-166.9</td>
<td>-166.9</td>
</tr>
<tr>
<td>( P_3^4 - P_1^4 )</td>
<td>1055</td>
<td>334.9</td>
<td>327.5</td>
<td>1003</td>
<td>1055</td>
<td>1055</td>
</tr>
</tbody>
</table>

cell analysis only, the proposed approach is very efficient and requires far less computational effort than approaches based on full FEM analysis. The present approach is valid for beams composed of repetitive cells along its span. The geometrical and material properties of the unit cell structures may be arbitrarily complex as long as the cell’s two end cross-sections are identical.

The achievements of this paper are twofold: (1) a novel method for predicting the stiffness of and recovering local stresses in heterogenous, anisotropic beams is developed based on the three-dimensional finite element analysis of a single unit cell and (2) a unified approach for the efficient, yet accurate static analysis for heterogenous beams composed of repetitive cells is proposed. The approach can be implemented using existing commercial finite element packages only.

A Symplectic matrices

Matrix \( S \), of size \( 2n \times 2n \), is said to be symplectic if it satisfies the following property

\[
S^T \mathcal{J} S = \mathcal{J},
\]

where skew-symmetric matrix \( \mathcal{J} \) is defined as

\[
\mathcal{J} = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{bmatrix}.
\]
Table 5: Axial forces of the beam components in the 5th cell, [N].

<table>
<thead>
<tr>
<th>Beam</th>
<th>ABAQUS</th>
<th>$n_e = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2^5 - P_3^4$</td>
<td>$-192.2$</td>
<td>$-192.2$</td>
</tr>
<tr>
<td>$P_2^5 - P_3^4$</td>
<td>$1109$</td>
<td>$1109$</td>
</tr>
<tr>
<td>$P_2^5 - P_3^4$</td>
<td>$-247.6$</td>
<td>$-247.6$</td>
</tr>
<tr>
<td>$P_2^5 - P_3^4$</td>
<td>$318$</td>
<td>$318$</td>
</tr>
<tr>
<td>$P_4^5 - P_3^4$</td>
<td>$-895.1$</td>
<td>$-895.1$</td>
</tr>
<tr>
<td>$P_4^5 - P_3^4$</td>
<td>$-46.57$</td>
<td>$-46.57$</td>
</tr>
<tr>
<td>$P_4^5 - P_3^4$</td>
<td>$-551.5$</td>
<td>$-551.5$</td>
</tr>
<tr>
<td>$P_3^5 - P_4^6$</td>
<td>$469.3$</td>
<td>$469.3$</td>
</tr>
<tr>
<td>$P_3^5 - P_4^6$</td>
<td>$48.78$</td>
<td>$48.78$</td>
</tr>
</tbody>
</table>

Table 6: Axial forces of the beam components in the 10th cell, [N].

<table>
<thead>
<tr>
<th>Beam</th>
<th>ABAQUS</th>
<th>$n_e = 0$</th>
<th>$n_e = 6$</th>
<th>$n_e = 16$</th>
<th>$n_e = 18$</th>
<th>$n_e = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2^{10} - P_1^{11}$</td>
<td>$15.49$</td>
<td>$14.45$</td>
<td>$15.45$</td>
<td>$16.27$</td>
<td>$15.49$</td>
<td>$15.49$</td>
</tr>
<tr>
<td>$P_2^{10} - P_1^{11}$</td>
<td>$1.200$</td>
<td>$5.942$</td>
<td>$4.483$</td>
<td>$1.476$</td>
<td>$1.199$</td>
<td>$1.199$</td>
</tr>
<tr>
<td>$P_2^{10} - P_1^{11}$</td>
<td>$85.56$</td>
<td>$83.78$</td>
<td>$84.40$</td>
<td>$84.60$</td>
<td>$85.56$</td>
<td>$85.56$</td>
</tr>
<tr>
<td>$P_2^{10} - P_1^{11}$</td>
<td>$-98.36$</td>
<td>$-98.94$</td>
<td>$-99.03$</td>
<td>$98.74$</td>
<td>$-98.36$</td>
<td>$-98.36$</td>
</tr>
<tr>
<td>$P_4^{10} - P_1^{11}$</td>
<td>$-55.41$</td>
<td>$-54.68$</td>
<td>$-55.64$</td>
<td>$-55.41$</td>
<td>$-55.41$</td>
<td></td>
</tr>
<tr>
<td>$P_4^{10} - P_1^{11}$</td>
<td>$43.02$</td>
<td>$44.98$</td>
<td>$44.78$</td>
<td>$42.90$</td>
<td>$43.02$</td>
<td>$43.02$</td>
</tr>
<tr>
<td>$P_4^{10} - P_1^{11}$</td>
<td>$169.4$</td>
<td>$166.0$</td>
<td>$166.7$</td>
<td>$169.9$</td>
<td>$169.4$</td>
<td>$169.4$</td>
</tr>
<tr>
<td>$P_4^{10} - P_1^{11}$</td>
<td>$82.48$</td>
<td>$85.08$</td>
<td>$81.87$</td>
<td>$82.48$</td>
<td>$82.48$</td>
<td></td>
</tr>
<tr>
<td>$P_4^{10} - P_1^{11}$</td>
<td>$7.139$</td>
<td>$-7.139$</td>
<td>$-7.139$</td>
<td>$-7.139$</td>
<td>$-7.139$</td>
<td></td>
</tr>
</tbody>
</table>

Note the following properties of matrix $J$: $J^T = -J$, $J J^T = I$.

Property (51) implies that the most general form of a symplectic matrix is

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where matrices $A$, $B$, $C$, and $D$, each of size $n \times n$, satisfy the following conditions

$$\begin{align*}
(A^T C)^T &= A^T C, \\
(B^T D)^T &= B^T D, \\
A^T D - C^T B &= I, \\
B^T C - D^T A &= -I.
\end{align*}$$

**B Symplectic transformations**

Two vectors, $\mathbf{x}$ and $\mathbf{y}$, both of size $2n \times 1$, are said to be symplectic orthogonal if they satisfy the following property $\mathbf{x}^T J \mathbf{y} = 0$. Matrix $\mathbf{u}$ of size $2n \times 2m$ ($m \leq n$), is said to be symplectic if it satisfies the following property

$$\mathbf{u}^T J \mathbf{u} = \hat{J},$$

where skew-symmetric matrix $\hat{J}$ is of size $2m \times 2m$ but otherwise identical to that defined by eq. (52). Each column of matrix $\mathbf{u}$ is symplectic orthogonal to its other columns.
Consider now the following transformation
\[ SU = U \hat{S}. \] 
(56)

Pre-multiplying this equation by \( U^T \) yields \( U^T (S^T J S) U = S^T U^T J U \hat{S} = \hat{S} J \hat{S} = J \) and hence, if matrix \( S \) is symplectic, so is matrix \( \hat{S} \). Clearly, transformation (56) is structure preserving: symplectic matrix \( S \), of size \( 2n \times 2n \), is reduced to symplectic matrix \( \hat{S} \), of size \( 2m \times 2m \), by symplectic matrix \( U \).

C Eigenvalues of symplectic matrices

Let \( \mu \) be an eigenvalue of a symplectic matrix associated with eigenvector \( U_\mu \), i.e., \( S U_\mu = \mu U_\mu \). It then follows that \( S^T J U_\mu = \mu S^T U_\mu \) and properties (51) then imply \( S^T (J U_\mu) = \mu^{-1} (J U_\mu) \). Clearly, if \( U_\mu \) is an eigenvector of \( S \) associated with eigenvalue \( \mu \), \( S^T U_\mu \) is an eigenvector of \( S^T \) associated with eigenvalue \( \mu^{-1} \). Because the spectra of eigenvalues of matrices \( S \) and \( S^T \) are identical, the eigenvalues of symplectic matrices occur in reciprocal pairs, \( \mu \) and \( \mu^{-1} \).

Let \( \mu \) and \( \lambda \) be two eigenvalues of a symplectic matrix and the associated eigenvectors are denoted \( U_\mu \) and \( U_\lambda \), respectively, i.e., \( S U_\mu = \mu U_\mu \) and \( S U_\lambda = \lambda U_\lambda \). Pre-multiplying the first equation by \( U_\lambda^T \) and the second by \( U_\mu^T \) leads to
\[
\begin{align*}
U_\lambda^T S^T J U_\mu &= \mu^{-1} U_\mu^T J U_\mu, \\
U_\mu^T J S U_\lambda &= \lambda U_\lambda^T J U_\lambda,
\end{align*}
\] 
(57a)

where \( S^T J S = S \) is used in the derivation of (57a). Addition of these two equations yields \((\mu^{-1} - \lambda) U_\mu^T J U_\lambda = 0\). If \( \lambda \mu \neq 1.0 \), the following symplectic orthogonality results
\[ U_\mu^T J U_\lambda = 0. \] 
(58)

For the reciprocal pairs, i.e., \( \lambda \mu = 1.0 \), eigenvectors \( U_\mu \) and \( U_{\mu^{-1}} \) can be normalized as \( U_\mu^T J U_{\mu^{-1}} = 1 \). In summary, let the columns of matrix \( \underline{U} \) store the eigenvectors associated with eigenvalues \( \mu \) and \( \mu^{-1} \), it follows that
\[ S \underline{U} = \underline{U} \text{diag}(\mu, \mu^{-1}). \] 
(59)

Matrix \( \underline{U} \) is symplectic
\[ \underline{U}^T J \underline{U} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \] 
(60)

References


